58. *On a Problem of R. Brauer on Zeta-Functions of Algebraic Number Fields. II*

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1. Let K_1, K_2 be algebraic number fields of finite degrees. Put $K=K_1K_2$, $k=K_1\cap K_2$ and consider the following quotient of Dedekind zeta-functions:

$$\zeta_{K_1,K_2}(s)=\zeta_K(s)\cdot\zeta_k(s)/\zeta_{K_1}(s)\cdot\zeta_{K_2}(s).$$

It was shown by R. Brauer [1] that $\zeta_{K_1,K_2}(s)$ is an entire function of s, if K_1/k and K_2/k are normal. In our previous note [2], we called *R. Brauer's problem* the question asking for other cases in which $\zeta_{K_1,K_2}(s)$ becomes entire. We proved that this takes place in the following cases:

(i) $K_1=Q(\sqrt[p]{a})$, $K_2=Q(\sqrt[p]{b})$, where p is an odd prime and a, b are relatively prime p-free integers $\neq 1$.

(ii) $K_1=Q(\sqrt[p]{a})$, $K_2=Q(\sqrt[q]{b})$ where p, q are distinct odd primes and a, b are relatively prime, respectively p-free and q-free integers $\neq 1$.

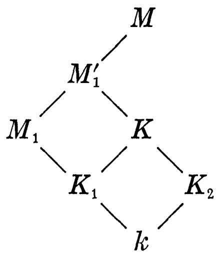
In the present note, we shall show that these results can be derived in a generalized form from a theorem on "supersolvable extensions" as stated below. The letters k, K, L, M (sometimes with suffixes) will denote throughout this note algebraic number fields of finite degrees.

2. If K/k is normal and Gal(K/k) is supersolvable, K/k itself will be called *supersolvable*. Then there exists a chain of intermediate fields $K=k_\nu\supset k_{\nu-1}\supset\cdots\supset k_0=k$ such that all k_i/k are normal and $k_i\supset k_{i-1}$ are cyclic, $i=\nu, \nu-1, \cdots, 1$. It is known that if K/k is supersolvable, the Artin L-function $L(s, \chi, K/k)$ for every non-principal character χ of Gal (K/k) is entire (cf. [3]).

Theorem. *Let $K=K_1K_2$, $k=K_1\cap K_2$. Let M/k, M_1/k be galois closures of K/k, K_1/k respectively. If M/k is supersolvable and $M_1\cap K_2=k$, then $\zeta_{K_1,K_2}(s)$ is entire.*

Proof. Put $G=\text{Gal}(M/k)$, $G_1=\text{Gal}(M_1/k)$, $H_1=\text{Gal}(M_1/K_1)$. Then we have after Artin $\zeta_{K_1}(s)=L(s, 1_{H_1}, M_1/K_1)=L(s, 1_{H_1}^{G_1}, M_1/k)$, where 1_{H_1} is the principal character of H_1 and $1_{H_1}^{G_1}$ the same character induced to G_1. Likewise $\zeta_k(s)=L(s, 1_{G_1}, M_1/k)$. Now we can write $1_{H_1}^{G_1}=1_{G_1}+\sum_i\lambda_i$, where λ_i are nonprincipal irreducible characters of G_1, so that we obtain

(1) $\zeta_{K_1}(s)/\zeta_k(s)=\prod_i L(s, \lambda_i, M_1/k)=\prod_i L(s, \tilde{\lambda}_i, M/k)$. Here $\tilde{\lambda}_i$ is the character λ_i lifted to Gal(M/k). We give the following diagram for the sake of convenience.



Put $M'_1=M_1K=M_1K_2$, then M'_1/K_2 is normal and $\mathrm{Gal}(M'_1/K_2)\cong\mathrm{Gal}(M_1/k)=G_1$ so that just as above

$$(2)\qquad \zeta_K(s)/\zeta_{K_2}(s)=\prod_i L(s,\lambda_i,M'_1/K_2)$$
$$=\prod_i L(s,\tilde{\lambda}_i,M/K_2)=\prod_i L(s,\tilde{\lambda}_i^G,M/k)$$

where $\tilde{\lambda}_i^G$ is the lifted character $\tilde{\lambda}_i$ induced to G, which can be written in the form $\tilde{\lambda}_i+\sum_j \lambda'_{ij}$, where λ'_{ij} are non-principal irreducible characters of G. Thus dividing (2) by (1), we see that $\zeta_{K_1,K_2}(s)$ is equal to a product of the form $\prod_{i,j} L(s,\lambda'_{ij},M/k)$ which is entire.

3. Now let m, n be any given natural numbers $\geqq 2$ and $a, b\in \boldsymbol{Z}$.

Lemma 1. *The galois closure K of $\boldsymbol{Q}(\sqrt[m]{a},\sqrt[n]{b})$ over $\boldsymbol{Q}$ is supersolvable.*

Proof. Let l be the L. C. M. of m, n, and put $\omega=\exp(2\pi i/l)$, $\boldsymbol{Q}(\omega)=L_0$, $\boldsymbol{Q}(\sqrt[m]{a},\sqrt[n]{b})=K_0$, $L_0K_0=L$, $\boldsymbol{Q}=k$. Then L/k is normal, $L\supset K\supset k$ and K/k is normal. It suffices clearly to show that L/k is supersolvable. Now $L\supset L_0\supset k$, L/L_0 is Kummerian and L_0/k is cyclotomic. So it is easy to construct a chain of intermediate fields $L=k_\nu\supset k_{\nu-1}\supset\cdots\supset k_\rho=L_0\supset k_{\rho-1}\supset\cdots\supset k_0=k$ such that k_i/k are normal and k_i/k_{i-1} are cyclic, $i=\nu,\nu-1,\cdots,1$.

For a prime p and $a\in\boldsymbol{Z}$, $v_p(a)$ denotes as usual the natural number such that $p^{v_p(a)}\|a$. If $(m, v_p(a))=1$, p will be called an *m-proper prime divisor* of a. The product of all m-proper prime divisors of a will be denoted by $(a)_m$. If $(a)_m\neq 1$, the degree of $\boldsymbol{Q}(\sqrt[m]{a})$ over $\boldsymbol{Q}$ is m and every m-proper prime divisor is completely ramified in $\boldsymbol{Q}(\sqrt[m]{a})$. The galois closure of $\boldsymbol{Q}(\sqrt[m]{a})$ (over $\boldsymbol{Q}$) is contained in $\boldsymbol{Q}(\sqrt[m]{a},\exp(2\pi i/m))$. The degree of this latter field divides $m\varphi(m)$, where φ is the Euler function and the only primes that can be ramified in it are divisors of ma. From these facts we obtain;

Lemma 2. *Suppose $(a)_m\neq 1$, $(b)_n\neq 1$ and put $K_1=\boldsymbol{Q}(\sqrt[m]{a})$, $K_2=\boldsymbol{Q}(\sqrt[n]{b})$, $K=K_1K_2$, $k=K_1\cap K_2$. If $(m, n)=1$ or $((a)_m,(b)_n)=1$, we have $k=\boldsymbol{Q}$, and if moreover $(ma,(b)_n)=1$ or $(m\varphi(m), n)=1$, we have $M_1\cap K_2=k$, where M_1/k is the galois closure of K_1/k.*

In virtue of these Lemmas our theorem yields the following Corollary from which our previous results (i), (ii) follow immediately.

Corollary. *Let $(a)_m\neq 1$, $(b)_n\neq 1$ and $(m, n)=1$ or $((a)_m,(b)_n)=1$. Then $\zeta_{K_1,K_2}(s)$ is entire, if $(ma,(b)_n)=1$ or $((a)_m, nb)=1$ or $(m\varphi(m), n)=1$ or $(m, n\varphi(n))=1$.*

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References

[1] R. Brauer: A note on zeta-functions of algebraic number fields. Acta Arith., **24**, 325–327 (1973).
[2] K. Sato: On a problem of R. Brauer on zeta-functions of algebraic number fields. Proc. Japan Acad., **61**A, 305–307 (1985).
[3] K. Uchida: On Artin L-functions. Tôhoku Math. J., **27**, 75–81 (1975).