53. *Whittaker Models for Highest Weight Representations of Semisimple Lie Groups and Embeddings into the Principal Series*

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Let G be a connected, real simple linear Lie group and K its maximal compact subgroup. Assume that G/K is a hermitian symmetric space. The aim of this note is to describe embeddings of irreducible highest weight G-modules, including the holomorphic discrete series and finite-dimensional representations, into two types of interesting induced representations: Kawanaka's generalized Gelfand-Graev representations (GGGRs) and the principal series.

1. Method and preparation. We employ here the method of highest weight vectors (cf. Hashizume [2]). Precisely, we determine all the K-finite highest weight vectors in GGGRs and the principal series by solving systems of differential equations characterizing such vectors. This enables us to describe embeddings of highest weight modules.

We prepare a refined structure theorem for $\mathfrak{g}\equiv\mathrm{Lie}(G)$, due to Moore. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ with $\mathfrak{k}\equiv\mathrm{Lie}(K)$, be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of G. Then there exists a unique central element Z_0 of $\mathfrak{k}$ such that $\mathrm{ad}(Z_0)|\mathfrak{p}$ gives the $\mathrm{Ad}(K)$-invariant complex structure on $\mathfrak{p}$ coming from the given G-invariant one on G/K. Putting $\mathfrak{p}_{\pm}=\{X\in\mathfrak{p}_C;\ [Z_0,X]=\pm\sqrt{-1}X\}$, one gets a decomposition $\mathfrak{g}_C=\mathfrak{p}_-\oplus\mathfrak{k}_C\oplus\mathfrak{p}_+$. Let $\mathfrak{t}\subseteq\mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, and Δ the root system of $(\mathfrak{g}_C,\mathfrak{t}_C)$. Select a positive system Δ^+ of Δ such as $\gamma(Z_0)=\sqrt{-1}$ for all non-compact positive roots γ. We denote by $\Delta_{\mathfrak{k}}^+$ (resp. $\Delta_{\mathfrak{p}}^+$) the set of compact (resp. non-compact) positive roots. Construct a sequence $(\gamma_1,\gamma_2,\cdots,\gamma_l)$ of non-compact positive roots inductively as follows: γ_k is the largest root of $\Delta_{\mathfrak{p}}^+$ strongly orthogonal to $\gamma_m:\gamma_k\pm\gamma_m\notin\Delta\cup\{0\}$, for all $m>k$. We take a root vector X_γ for a $\gamma\in\Delta$ satisfying

$$X_\gamma-X_{-\gamma},\quad \sqrt{-1}(X_\gamma+X_{-\gamma})\in\mathfrak{k}+\sqrt{-1}\mathfrak{p},\quad [X_\gamma,X_{-\gamma}]=H'_\gamma.$$

Here, $H'_\gamma\equiv 2H_\gamma/\gamma(H_\gamma)$ with the element $H_\gamma\in\sqrt{-1}\mathfrak{t}$ determined by $\gamma(H)=B(H,H_\gamma)$ $(H\in\mathfrak{t}_C)$, through the Killing form B of $\mathfrak{g}_C$.

We put $H_k=X_{\gamma_k}+X_{-\gamma_k}\in\mathfrak{p}$ for $1\leqq k\leqq l$. Then, $\mathfrak{a}_p\equiv\sum_{1\leqq k\leqq l}\boldsymbol{R}H_k$ is a maximal abelian subspace of $\mathfrak{p}$. Let μ be a Cayley transform of $\mathfrak{g}_C$ defined by $\mu=\exp((\pi/4)\cdot\sum_{1\leqq k\leqq l}\mathrm{ad}(X_{\gamma_k}-X_{-\gamma_k}))$. Then, $\mu(H_k)=H'_{\gamma_k}$, whence $\psi_k\equiv(\gamma_k/2)\circ(\mu|\mathfrak{a}_p)$ $(1\leqq k\leqq l)$ form an orthogonal basis of $\mathfrak{a}_p^*$, the dual space of $\mathfrak{a}_p$. The root system Ψ of $(\mathfrak{g},\mathfrak{a}_p)$ is related with Δ via $(\Delta\circ(\mu|\mathfrak{a}_p))\cup\{0\}=\Psi\cup\{0\}$. We select a positive system Ψ^+ of Ψ consistent with $\Delta^+\subseteq\Delta$ under this relation. According to Moore [3], there are only two possibilities for Ψ^+:

(I) $\Psi^+\cup\{0\}=\{\psi_k\pm\psi_m;\ 1\leqq m\leqq k\leqq l\}$ if G/K is of tube type,

(II) $\Psi^+\cup\{0\}=\{\psi_k\pm\psi_m;\ 1\leqq m\leqq k\leqq l\}\cup\{\psi_k;\ 1\leqq k\leqq l\}$ otherwise.

We set $\mathfrak{n}_0=\sum_{k>m}\mathfrak{g}_{\psi_k-\psi_m}$, $\mathfrak{n}_1=\sum_k\mathfrak{g}_{\psi_k}$ (possibly zero), $\mathfrak{n}_2=\sum_{k\geqq m}\mathfrak{g}_{\psi_k+\psi_m}$, $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$ and $\mathfrak{n}_m=\mathfrak{n}_0+\mathfrak{n}$. Here, $\mathfrak{g}_\psi$ denotes the root space for $\psi\in\Psi$. Note that $\mathfrak{n}_2=\mathfrak{z}_\mathfrak{n}$, the center of $\mathfrak{n}$. Denote by N_0, N and N_m the analytic subgroups of G with Lie algebras $\mathfrak{n}_0$, $\mathfrak{n}$ and $\mathfrak{n}_m$ respectively. Then one gets $G=KA_pN_m$ (an Iwasawa decomposition) with $A_p=\exp\mathfrak{a}_p$, and $N_m=N_0N=N_0\ltimes N$, a semidirect product, where N is normal. The nilpotent Lie group N is at most two-step, and abelian exactly in the case (I).

2. Highest weight modules (cf. [5, § 2]). Let Ξ be the set of linear forms λ on $\mathfrak{t}_C$ satisfying (1) $\lambda(H'_\gamma)\geqq 0$ for all $\gamma\in\Delta_\mathfrak{t}^+$, and (2) $\exp H\mapsto\exp\lambda(H)$ ($H\in\mathfrak{t}$), gives a unitary character of the maximal torus $\exp\mathfrak{t}\subseteqq K$. For each $\lambda\in\Xi$, there exists a unique (up to equivalence) irreducible admissible $(\mathfrak{g}_C, K)$-module L_λ with Δ^+-highest weight λ. Further, each L_λ globalizes to a continuous irreducible representation π_λ of G on a Hilbert space for which the $(\mathfrak{g}_C, K)$-module of K-finite vectors is isomorphic to L_λ. If $\lambda\in\Xi$ satisfies

(2.1) $$(\lambda+\rho)(H'_\gamma)<0 \quad \text{for all } \gamma\in\Delta_\mathfrak{p}^+\ (\rho\equiv(1/2)\cdot\textstyle\sum_{\gamma\in\Delta^+}\gamma),$$

then L_λ is unitarizable and π_λ belongs to the holomorphic discrete series. On the other hand, L_λ is finite-dimensional if λ is dominant with respect to the whole Δ^+.

3. GGGRs Γ_i. Let i be an integer such that $0\leqq i\leqq l$. We put $A[i]=-\sum_{k\leqq i}E_k+\sum_{m>i}E_m$, where $E_k\equiv-\sqrt{-1}\mu^{-1}(X_{\gamma_k})\in\mathfrak{g}$ is a root vector for $2\psi_k$. The GGGR associated with the nilpotent class $\mathrm{Ad}(G)A[i]$ of $\mathfrak{g}$ is defined to be an induced representation $\Gamma_i=\mathrm{Ind}_N^G(\xi_i)$ (cf. [6]). Here, ξ_i is an irreducible unitary representation of N corresponding to the linear form $A[i]^*:\mathfrak{n}\ni Z\mapsto B(Z,\theta A[i])$, through the Kirillov orbit method.

To describe highest weight vectors for Γ_i, we realize ξ_i explicitly on a Fock space. Set $V_k^\pm=(\mathfrak{g}_{\psi_k})_C\cap(\mathfrak{k}_C+\mathfrak{p}_\pm)$ for $1\leqq k\leqq l$, then one gets $(\mathfrak{g}_{\psi_k})_C=V_k^+\oplus V_k^-$ and $\overline{V_k^+}=V_k^-$. Here the bar means the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. So there exists a unique complex structure J_i on $\mathfrak{n}_1$ such that the $(\sqrt{-1})$-eigenspace for the complex linear extension of J_i to $(\mathfrak{n}_1)_C$ coincides with $\sum_{k\leqq i}V_k^-+\sum_{m>i}V_m^+$. The quadratic form: $U\mapsto -A[i]^*([J_iU,U])/4$, on $\mathfrak{n}_1$ is positive definite, and it induces canonically a hermitian inner product $(\ ,\ )_i$ on the complex vector space $(\mathfrak{n}_1, J_i)$.

Let $\mathcal{F}_i$ be the Fock space of $(\mathfrak{n}_1, J_i, (\ ,\ )_i)$, which consists of entire functions ϕ on $(\mathfrak{n}_1, J_i)$ satisfying

(3.1) $$\|\phi\|_i^2\equiv\int_{\mathfrak{n}_1}|\phi(U)|^2\exp(-2(U,U)_i)dU<\infty,$$

where dU is a Lebesgue measure on $\mathfrak{n}_1$. We can realize ξ_i on $\mathcal{F}_i$ as

(3.2) $$\xi_i(n)\phi(U)=\exp\{(2U-X_1,X_1)_i+\sqrt{-1}A[i]^*(X_2)\}\cdot\phi(-X_1+U)$$

for $n=\exp(X_1+X_2)\in N$ with $X_j\in\mathfrak{n}_j$.

4. Highest weight vectors for Γ_i. We determine explicitly all the K-finite highest weight vectors for GGGR Γ_i in C^∞- or L^2-context.

Now let $G_C \supseteqq G$ be the complexification of G, and K_C the analytic subgroup of G_C with Lie algebra $\mathfrak{k}_C$. Denote by α the restriction of $\mu\circ\theta$ to a solvable Lie subalgebra $\mathfrak{a}_p+\mathfrak{n}_0+V^+\subseteqq\mathfrak{g}_C$ with $V^+\equiv\sum_k V_k^+$. Then α is lifted up canonically to a group isomorphism from $A_pN_0\exp V^+$ into K_C, denoted still by α. For $\lambda\in\Xi$, let $(\tau_\lambda, V_\lambda)$ be an irreducible holomorphic representation of K_C with highest weight λ, and $(\tau_\lambda^*, V_\lambda^*)$ its contragredient.

We first consider the C^∞-induced GGGR C^∞-Γ_i acting on the space:

(4.1) $\quad C^\infty(G;\xi_i)\equiv\{F: G\xrightarrow{C^\infty}\mathcal{H}_i;\ F(gn)=\xi_i(n)^{-1}F(g)\ (g\in G,\ n\in N)\}$,

by left translation. By differentiating this G-action, $C^\infty(G;\xi_i)$ has a $\mathfrak{g}_C$-module structure. Since $\mathcal{H}_i$ consists of functions on $\mathfrak{n}_1$, each $F\in C^\infty(G;\xi_i)$ is viewed canonically as a function $(g, U)\mapsto F(g:U)$ on $G\times\mathfrak{n}_1$. For a $\lambda\in\Xi$, let $Y_i(\lambda)$ denote the space of K-finite, Δ^+-highest weight vectors in $C^\infty(G;\xi_i)$ with highest weight λ.

Theorem 1. (1) $Y_i(\lambda)=(0)$ *if G/K is of non-tube type and $i\neq 0$.*

(2) *Assume that G/K is of tube type or $i=0$. For a $v^*\in V_\lambda^*$, put*

$$F_{v^*}^{\lambda i}(kan_0:U)=(\exp\langle A[i]^*, \mathrm{Ad}(an_0)^{-1}A[0]\rangle)\cdot\langle v_\lambda, \tau_\lambda^*(k\alpha(an_0\exp\tilde{U}))v^*\rangle,$$

where $(k, a, n_0)\in K\times A_p\times N_0$, $\tilde{U}\equiv(U-\sqrt{-1}J_0U)/2\sqrt{2}\in V^+$ $(U\in\mathfrak{n}_1)$, and v_λ is a non-zero highest weight vector for τ_λ. Then, $F_{v^*}^{\lambda i}$ extends uniquely to an element of $Y_i(\lambda)$ through the relation in (4.1). *Moreover, the map from v^* to the extended $F_{v^*}^{\lambda i}$ gives an isomorphism of vector spaces: $V_\lambda^*\simeq Y_i(\lambda)$, for every $\lambda\in\Xi$.*

Next, by evaluating L^2-norm, we can specify highest weight vectors $F_{v^*}^{\lambda i}$ contained in the space of unitarily induced GGGR L^2-$\Gamma_i\equiv L^2$-$\mathrm{Ind}_N^G(\xi_i)$.

Theorem 2. *Under the assumption of Theorem* 1(2), *the $\mathcal{H}_i$-valued function $F_{v^*}^{\lambda i}$ on G is square-integrable modulo N if and only if $i=0$ and λ satisfies* (2.1).

5. Whittaker models in GGGRs. Thanks to Theorems 1 and 2, we can describe embeddings, or Whittaker models, of L_λ's into GGGRs Γ_i.

Theorem 3 (C^∞-case). *Let $\lambda\in\Xi$. Then, L_λ is embedded into C^∞-Γ_i only if G/K is of tube type or $i=0$. In this case, its multiplicity as submodules is bounded by* $\dim\tau_\lambda$. *Furthermore, for λ such that $(\lambda+\rho)(H'_\gamma)\leqq 0$ $(\gamma\in\Delta_p^+)$, the highest weight module L_λ, which corresponds to the holomorphic discrete series or its limit, occurs in C^∞-Γ_i exactly* $\dim\tau_\lambda$ *times.*

Theorem 4 (L^2-case). *The representation π_λ occurs in L^2-Γ_i as its subrepresentation if and only if $i=0$ and π_λ lies in the holomorphic discrete series. In this case its multiplicity equals* $\dim\tau_\lambda$.

Remark 1. Let $S\equiv A_pN_m$ be an Iwasawa subgroup of G. Through the Frobenius reciprocity for (G, S), we can derive from Theorem 4 Rossi-Vergne's result [4, Corollary 5.23] which describes the restriction of holomorphic discrete series to the subgroup S. Nevertheless, our proof of Theorem 4 is independent of their result.

6. Embeddings into the principal series. Let $P_m=MA_pN_m$ be a minimal parabolic subgroup of G, where M is the centralizer of A_p in K. For an irreducible representation σ of M and $\psi \in (\mathfrak{a}_p^*)_C$, consider the C^∞-induced representation, called of principal series, $\pi_{\sigma,\psi} \equiv C^\infty\text{-Ind}_{P_m}^G(\sigma \otimes e^\psi \otimes 1_{N_m})$ defined as in (4.1), where 1_{N_m} denotes the trivial character of N_m. By determining all the K-finite highest weight vectors in the principal series, we obtain a complete description of embeddings of L_λ's into $\pi_{\sigma,\psi}$'s as follows.

Theorem 5. *For every $\lambda \in \Xi$, the irreducible λ-highest weight $(\mathfrak{g}_C, K)$-module L_λ is embedded into the uniquely determined principal series $\pi_{\sigma_\lambda,\psi_\lambda}$ with multiplicity one. Here, σ_λ denotes the irreducible representation of M acting on the M-submodule of V_λ generated by the highest weight vector v_λ, and $\psi_\lambda \equiv (-\lambda) \circ (\mu | \mathfrak{a}_p) \in \mathfrak{a}_p^*$.*

Remark 2. Through the compact picture of the principal series, one can construct a continuous representation of G on a Hilbert space $H_{\sigma,\psi}$ such that the corresponding smooth representation on the space $H_{\sigma,\psi}^\infty \subseteq H_{\sigma,\psi}$ of C^∞-vectors is equivalent to $\pi_{\sigma,\psi}$. Then, the closure of the image of embedding $L_\lambda \hookrightarrow H_{\sigma_\lambda,\psi_\lambda}^\infty$ in $H_{\sigma_\lambda,\psi_\lambda}$ is an irreducible G-submodule of $H_{\sigma_\lambda,\psi_\lambda}$ with highest weight λ.

Remark 3. Collingwood [1] obtained the unique embedding property without specifying the place where L_λ can be embedded.

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