51. *The Steffensen Iteration Method for Systems of Nonlinear Equations.* II

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1. **Introduction.** In generalizing the Aitken δ^2-process in one dimension to the case of n-dimensions, Henrici [1, p. 116] has considered a formula, which is called the Aitken-Steffensen formula. In [2], we have studied the above Aitken-Steffensen formula for systems of nonlinear equations and shown [2, Theorem 2]. Moreover, in [3], we have considered a method of iteration for the above systems, which is often called the Steffensen iteration method, and shown [3, Theorem 1]. [3, Theorem 1] improves the result of [2, Theorem 2].

We have given the proof of [3, Theorem 1], in which the Sherman-Morrison-Woodbury formula [3, Lemma 4] is used only to determine $(\Delta^2 X(x^{(k)}))^{-1}$, but in this paper we show that the proof can be simplified without using the formula. And we also present a numerical example in order to show the efficiency of the Steffensen iteration method.

2. **Statement of results.** Let $x=(x_1, x_2, \cdots, x_n)$ be a vector in R^n and D a region contained in R^n. Let $f_i(x)$ $(1\leqq i\leqq n)$ be real-valued nonlinear functions defined on D and $f(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ an n-dimensional vector-valued function. Then we shall consider a system of nonlinear equations

$$x=f(x), \tag{2.1}$$

whose solution is $\bar{x}$. Let $\|x\|$ and $\|A\|$ be denoted by

$$\|x\|=\max_{1\leqq i\leqq n}|x_i| \quad \text{and} \quad \|A\|=\max_{1\leqq i\leqq n}\sum_{j=1}^{n}|a_{ij}|,$$

where $A=(a_{ij})$ is an $n\times n$ matrix. Define $f^{(i)}(x)\in R^n$ $(i=0, 1, 2, \cdots)$ by

$$f^{(0)}(x)=x,$$
$$f^{(i)}(x)=f(f^{(i-1)}(x)) \qquad (i=1, 2, \cdots).$$

Put

$$d^{(0,k)}=x^{(k)}-\bar{x},$$
$$d^{(i,k)}=f^{(i)}(x^{(k)})-\bar{x} \qquad \text{for } i=1, 2, \cdots,$$

and then define an $n\times n$ matrix $D(x^{(k)})$ by

$$D(x^{(k)})=(d^{(0,k)}, d^{(1,k)}, \cdots, d^{(n-1,k)}).$$

Throughout this paper, we shall assume the following five conditions (A.1)–(A.5) which are the same as those of [3].

(A.1) $f_i(x)$ $(1\leqq i\leqq n)$ are two times continuously differentiable on D.

(A.2) There exists a point $\bar{x}\in D$ satisfying (2.1).

(A.3) $\|J(\bar{x})\|<1$, where $J(x)=(\partial f_i(x)/\partial x_j)$ $(1\leqq i, j\leqq n)$.

(A.4) The vectors $d^{(0,k)}, d^{(1,k)}, \cdots, d^{(n-1,k)}$, $k=0,1,2,\cdots$, are linearly independent.

(A.5) $\inf\{|\det D(x^{(k)})|/\|d^{(0,k)}\|^n\}>0$.

Now, we consider Steffensen's iteration method

$$x^{(k+1)}=x^{(k)}-\Delta X(x^{(k)})(\Delta^2 X(x^{(k)}))^{-1}\Delta x(x^{(k)}), \tag{2.2}$$

where an n-dimensional vector $\Delta x(x)$, and $n\times n$ matrices $\Delta X(x)$ and $\Delta^2 X(x)$ are given by

$$\Delta x(x)=f^{(1)}(x)-x,$$
$$\Delta X(x)=(f^{(1)}(x)-x, \cdots, f^{(n)}(x)-f^{(n-1)}(x))$$

and

$$\Delta^2 X(x)=(f^{(2)}(x)-2f^{(1)}(x)+x, \cdots, f^{(n+1)}(x)-2f^{(n)}(x)+f^{(n-1)}(x)).$$

In this paper, we also show the following

Theorem 1. *Under conditions* (A.1)–(A.5), *there exists a constant M such that an estimate of the form*

$$\|x^{(k+1)}-\bar{x}\|\leqq M\|x^{(k)}-\bar{x}\|^2$$

holds, provided that the $x^{(k)}$ generated by (2.2) *are sufficiently close to the solution $\bar{x}$ of* (2.1).

3. Preliminaries. For the proof of Theorem 1, we need the following three lemmas given in [3]:

Lemma 1 ([3, Lemma 1]). *Let A and C be $n\times n$ matrices and assume that A is invertible, with $\|A^{-1}\|\leqq K_1$. If $\|A-C\|\leqq K_2$ and $K_1K_2<1$, then C is also invertible, and $\|C^{-1}\|\leqq K_1/(1-K_1K_2)$.*

Lemma 2 ([3, Lemma 2]). *Under conditions* (A.1)–(A.5), *there exists a constant L_1 such that the inequality*

$$\|(D(x^{(k)}))^{-1}\|\leqq L_1\|d^{(0,k)}\|^{-1}$$

holds for $x^{(k)}$ sufficiently close to $\bar{x}$.

Lemma 3 ([3, Lemma 3]). *Under conditions* (A.1)–(A.5), *$n\times n$ matrices $\Delta X(x^{(k)})$ and $\Delta^2 X(x^{(k)})$ are invertible, and there exist constants L_4 and L_7 such that the inequalities*

$$\|(\Delta X(x^{(k)}))^{-1}\|\leqq L_4\|d^{(0,k)}\|^{-1}, \tag{3.1}$$
$$\|(\Delta^2 X(x^{(k)}))^{-1}\|\leqq L_7\|d^{(0,k)}\|^{-1} \tag{3.2}$$

hold for $x^{(k)}$ sufficiently close to $\bar{x}$.

Lemmas 1 and 2 are used in proving Lemma 3. By the definition, we have

$$\Delta X(x^{(k)})=(J(\bar{x})-I)D(x^{(k)})+Y_1(x^{(k)}), \tag{3.3}$$
$$\Delta^2 X(x^{(k)})=(J(\bar{x})-I)\Delta X(x^{(k)})+Y_2(x^{(k)}), \tag{3.4}$$

where $Y_1(x)$ and $Y_2(x)$ are $n\times n$ matrices. By (A.1)–(A.3), we may choose constants L_2 and L_5 such that, for $x^{(k)}$ sufficiently close to $\bar{x}$,

$$\|Y_1(x^{(k)})\|\leqq L_2\|d^{(0,k)}\|^2, \tag{3.5}$$
$$\|Y_2(x^{(k)})\|\leqq L_5\|d^{(0,k)}\|^2. \tag{3.6}$$

Here we note that the inequality (3.1) holds with $L_4=L_1/L_3$ by choosing a constant L_3 so as to satisfy

$$1-\|J(\bar{x})\|-L_1L_2\|d^{(0,k)}\|\geqq L_3>0.$$

Similarly we obtain the inequality (3.2) with $L_7=L_4/L_6$ by choosing a constant L_6 satisfying

$$1-\|J(\bar{x})\|-L_4L_5\|d^{(0,k)}\|\geqq L_6>0.$$

4. Proof of Theorem 1. We shall prove Theorem 1. By the definition and (A.1)–(A.3), we also have, as in § 3,

$$\Delta x(x^{(k)})=(J(\bar{x})-I)d^{(0,k)}+\xi(x^{(k)}), \tag{4.1}$$

where $\xi(x)$ is an n-dimensional vector and

$$\|\xi(x^{(k)})\|\leqq L_8\|d^{(0,k)}\|^2, \tag{4.2}$$

a constant L_8 being suitably chosen.

We observe that, by Lemma 3, $\Delta X(x^{(k)})$ is invertible for $x^{(k)}$ sufficiently close to $\bar{x}$. Then, by (3.4),

$$J(\bar{x})-I=(\Delta^2X(x^{(k)})-Y_2(x^{(k)}))(\Delta X(x^{(k)}))^{-1}. \tag{4.3}$$

Substituting (4.1) into (2.2) and using (4.3), it yields

$$x^{(k+1)}-\bar{x}=\Delta X(x^{(k)})(\Delta^2X(x^{(k)}))^{-1}[Y_2(x^{(k)})\cdot(\Delta X(x^{(k)}))^{-1}d^{(0,k)}-\xi(x^{(k)})]. \tag{4.4}$$

Since $\|D(x^{(k)})\|\leqq\sum_{i=0}^{n-1}\|d^{(i,k)}\|$, we have

$$\|D(x^{(k)})\|\leqq\left(\sum_{i=0}^{n-1}M_1^i\right)\|d^{(0,k)}\|,$$

and so, from (3.3), by (A.3) and (3.5),

$$\|\Delta X(x^{(k)})\|\leqq L_9\|d^{(0,k)}\| \tag{4.5}$$

for a constant L_9 chosen suitably. In the above, we have used, under conditions (A.1)–(A.3), the fact that

$$\|d^{(i+1,k)}\|\leqq M_1\|d^{(i,k)}\| \qquad (0<M_1<1)$$

for $i=0, 1, 2, \cdots$. Hence, we obtain an estimate

$$\|x^{(k+1)}-\bar{x}\|\leqq L_9L_7(L_5L_4+L_8)\|x^{(k)}-\bar{x}\|^2, \tag{4.6}$$

from (4.4), by (4.5), (3.2), (3.6), (3.1) and (4.2). Therefore, (4.6) shows that Theorem 1 holds with $M=L_7L_9(L_4L_5+L_8)$. In this way, we have proved Theorem 1, as desired.

5. Numerical example. In order to show the efficiency of the Steffensen iteration method (2.2), we consider a system of nonlinear equations, Example 5.1, which is a modification of [4, (A.82)]. The solution of Example 5.1 using the Steffensen iteration method (2.2) is presented in Table 5.1 below, together with the solutions by the iteration method [2, (1.2)] and the Aitken-Steffensen formula [2, (1.5)].

Example 5.1.
$$\begin{cases} x_1=f_1(x_1, x_2)=\dfrac{1}{60}(3x_1^3-3x_1^2x_2+6x_1x_2^2+61.488), \\ x_2=f_2(x_1, x_2)=\dfrac{1}{50}(-x_1^3+6x_1^2x_2+3x_2^3-32.496). \end{cases}$$

The solution is $\bar{x}=(\bar{x}_1, \bar{x}_2)=(1.4, -1.0)$.

Table 5.1. Computation results for Example 5.1

Methods	Solutions
Iteration method [2, (1.2)]	$x^{(62)}=(1.3999000, -0.9999053)$
Aitken-Steffensen formula [2, (1.5)]	$y^{(32)}=(1.3999820, -0.9999861)$
Steffensen iteration method (2.2)	$x^{(4)}=(1.3999920, -0.9999936)$

$x^{(0)}=(0.0, 0.0)$

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