38. *Some Remarks on C-semigroups and Integrated Semigroups*

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1. Introduction. Let X be a Banach space and let $B(X)$ be the set of all bounded linear operators from X into itself. Let C be an injective operator in $B(X)$ and the range $R(C)$ be dense in X. According to Davies and Pang [3], a family $\{S(t);t\geq 0\}$ in $B(X)$ is called *C-semigroup*, if

(1.1) $S(t+s)C=S(t)S(s)$ for $t, s\geq 0$, and $S(0)=C$,

(1.2) $S(\cdot)x$: $[0, \infty)\to X$ is continuous for $x\in X$,

(1.3) there are $M\geq 0$ and $a\in \boldsymbol{R}\equiv(-\infty, \infty)$ such that $\|S(t)\|\leq Me^{at}$ for $t\geq 0$.

We define an operator G by $Gx=\lim_{t\to 0+}(C^{-1}S(t)x-x)/t$ for $x\in D(G)$ $\equiv\{x\in R(C); \lim_{t\to 0+}(C^{-1}S(t)x-x)/t$ exists$\}$. It is known that G is densely defined and closable, $\lambda-\overline{G}$ is injective for $\lambda>a$ and

(1.4) $(\lambda-\overline{G})\int_0^\infty e^{-\lambda t}S(t)x\,dt=Cx$ for $x\in X$ and $\lambda>a$.

(See [3], [4].) The closure $\overline{G}$ is called the *C-c.i.g.* of $\{S(t);t\geq 0\}$.

Let n be a positive integer. A family $\{U(t);t\geq 0\}$ in $B(X)$ is called *n-times integrated semigroup* (see [2]), if

(1.5) $U(\cdot)x: [0, \infty)\to X$ is continuous for $x\in X$,

(1.6) $U(t)U(s)x=\dfrac{1}{(n-1)!}\left(\int_t^{s+t}(s+t-r)^{n-1}U(r)x\,dr-\int_0^s(s+t-r)^{n-1}U(r)x\,dr\right)$

for $x\in X$ and $s, t\geq 0$, and $U(0)=0$,

(1.7) $U(t)x=0$ for all $t>0$ implies $x=0$,

(1.8) there are $M\geq 0$ and $\omega\in \boldsymbol{R}$ such that $\|U(t)\|\leq Me^{\omega t}$ for $t\geq 0$.

For convenience we call a C_0-semigroup also *0-times integrated semigroup.*

It is known [2] that if $\{U(t);t\geq 0\}$ is an n-times integrated semigroup, then there exists a unique closed linear operator A such that $(\omega, \infty)\subset\rho(A)$ (the resolvent set of A) and

(1.9) $R(\lambda;A)x(\equiv(\lambda-A)^{-1}x)=\int_0^\infty\lambda^n e^{-\lambda t}U(t)x\,dt$ for $x\in X$ and $\lambda>\omega$.

The operator A is called the *generator* of $\{U(t);t\geq 0\}$.

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let A be a densely defined closed linear operator in X with $\rho(A)\neq\emptyset$. Let $c\in\rho(A)$ and $n\geq 0$ be an integer. The following* (i)–(iii) *are equivalent:*

(i) *A is the generator of an n-times integrated semigroup $\{U(t);t\geq 0\}$.*

(ii) *A is the C-c.i.g. of a C-semigroup $\{S(t);t\geq 0\}$ with $C=R(c;A)^n$.*

(iii) *There exist* $M\geq 0$ *and* $a\in \boldsymbol{R}$ *such that* $(a,\infty)\subset\rho(A)$ *and* $\|R(\lambda;A)^m R(c;A)^n\|\leq M/(\lambda-a)^m$ *for* $m\geq 1$ *and* $\lambda>a$.

In this case, we have

$$U(t)x=(c-A)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}S(t_n)x\,dt_n\cdots dt_2dt_1 \text{ for } x\in X \text{ and } t\geq 0. \tag{1.10}$$

As a direct consequence of Theorem 1 we have

Theorem 2 ([1]). *Let E be an ordered Banach space whose positive cone is generating and normal. If A is a densely defined closed linear operator in E such that* $(\omega,\infty)\subset\rho(A)$ *for some* $\omega\in\boldsymbol{R}$ *and* $R(\lambda;A)\geq 0$ *for* $\lambda>\omega$ (i.e., *A is resolvent positive*), *then A is the generator of a once integrated semigroup* $\{U(t);t\geq 0\}$ *satisfying* $0\leq U(s)\leq U(t)$ *for* $0\leq s\leq t$.

Remarks. 1) The case $n=0$ in Theorem 1 is the Hille-Yosida theorem. 2) A in Theorem 1 (ii) coincides with the generator of $\{S(t);t\geq 0\}$ in the sense of [3]. 3) If $\sup_{x\neq 0,x\in R(C)}\|(c-A)^nS(t)x\|/\|x\|$ is bounded on $[\varepsilon,1/\varepsilon]$ for $\varepsilon\in(0,1)$, then $\{(c-A)^nS(t);t\geq 0\}$ becomes a semigroup (see [5, Theorem 2.2]). So, roughly speaking, (1.10) means that $U(t)$ may be represented as n-times integral of the semigroup $(c-A)^nS(t)$, $t\geq 0$. 4) Our proof of Theorem 2 seems to be simpler than the method in [1]. 5) In Theorem 2, it is shown that $R(\lambda;A)=\int_0^\infty e^{-\lambda t}dS(t)=\int_0^\infty \lambda e^{-\lambda t}S(t)\,dt$ for $\lambda>s(A)\equiv\inf\{\omega\in\boldsymbol{R};(\omega,\infty)\subset\rho(A)$ and $R(\lambda;A)\geq 0$ for $\lambda>\omega\}$ (see [1, Theorem 4.1]).

2. Proof of Theorems.

We start with the following

Lemma. *Let A be the C-c.i.g. of a C-semigroup* $\{S(t);t\geq 0\}$ *with* $C=R(c;A)^n$, *where* $c\in\rho(A)$ *and n is a positive integer, and let* $\|S(t)\|\leq Me^{at}$ *for* $t\geq 0$, *where* $M\geq 0$ *and* $a>0$ *are constants. Define* $V_k(t)$, $k\geq 0$, *by* $V_0(t)=S(t)$ *and* $V_k(t)x=\int_0^t\cdots\int_0^{t_{k-1}}S(t_k)x\,dt_k\cdots dt_1$ *for* $x\in X$ *and* $t\geq 0$.

Then for $k=1,2,\cdots,n$, *we have*

(2.1) $V_k(t)x\in D(A^k)$ *and* $\int_0^t(c-A)^{k-1}V_{k-1}(s)x\,ds\in D(A)$ *for* $x\in X$ *and* $t\geq 0$,

(2.2) $(c-A)^kV_k(t)\in B(X)$ *and* $\|(c-A)^kV_k(t)\|\leq M_ke^{at}$ *for* $t\geq 0$, *where* $M_k\geq 0$ *is a constant,*

(2.3) $(c-A)^kV_k(\cdot)x:[0,\infty)\to X$ *is continuous for* $x\in X$,

(2.4) $(c-A)^kV_k(t)=c(c-A)^{k-1}V_k(t)-(c-A)^{k-1}V_{k-1}(t)+(t^{k-1}/(k-1)!)\times(c-A)^{k-1}C$ *for* $t\geq 0$.

Proof. By (2.1) in [6]

$$\int_0^t S(s)x\,ds\in D(A) \text{ and } S(t)x-Cx=A\int_0^t S(s)x\,ds \text{ for } x\in X \text{ and } t\geq 0, \tag{2.5}$$

and hence (2.1)–(2.4) hold for $k=1$. The conclusion follows from induction with respect to k. Q.E.D.

Proof of Theorem 1. By virtue of [6, Theorem 2.1], (iii) implies (ii). To show that (ii) implies (i), let A be the C-*c.i.g.* of a C-semigroup $\{S(t);t\geq 0\}$ with $C=R(c;A)^n$ and $\|S(t)\|\leq Me^{at}$ for $t\geq 0$, where $a>0$. Define $U(t)$, $t\geq 0$, by $U(t)x=(c-A)^nV_n(t)x$ for $x\in X$. By Lemma, $U(t)\in B(X)$ for $t\geq 0$ and $\{U(t);t\geq 0\}$ satisfies (1.5), (1.7) and (1.8). Since

$$\int_0^\infty \lambda^k e^{-\lambda t}(c-A)^{k-1}V_k(t)x\,dt=\int_0^\infty \lambda^{k-1}e^{-\lambda t}(c-A)^{k-1}V_{k-1}(t)x\,dt,$$

we see from (2.4) that

$$\int_0^\infty \lambda^k e^{-\lambda t}(c-A)^k V_k(t)x\,dt=(c-\lambda)\int_0^\infty \lambda^{k-1}e^{-\lambda t}(c-A)^{k-1}V_{k-1}(t)x\,dt+(c-A)^{k-1}Cx,$$

i.e., $\int_0^\infty \lambda^k e^{-\lambda t}(c-A)^k V_k(t)x\,dt/(c-\lambda)^k$

$$=\int_0^\infty \lambda^{k-1}e^{-\lambda t}(c-A)^{k-1}V_{k-1}(t)x\,dt/(c-\lambda)^{k-1}+(c-A)^{k-1}Cx/(c-\lambda)^k$$

for $x\in X$, $1\leq k\leq n$ and $\lambda>a$. This implies that

$$\int_0^\infty \lambda^n e^{-\lambda t}U(t)x\,dt=\int_0^\infty \lambda^n e^{-\lambda t}(c-A)^n V_n(t)x\,dt=(c-\lambda)^n\int_0^\infty e^{-\lambda t}S(t)x\,dt + \textstyle\sum_{k=0}^{n-1}(c-\lambda)^k R(c\,;A)^{k+1}x.$$

Hence

$$(\lambda-A)\int_0^\infty \lambda^n e^{-\lambda t}U(t)x\,dt=(c-\lambda)^n(\lambda-A)\int_0^\infty e^{-\lambda t}S(t)x\,dt + \textstyle\sum_{k=0}^{n-1}((c-\lambda)^k R(c\,;A)^k x-(c-\lambda)^{k+1}R(c\,;A)^{k+1}x)=x$$

for $x\in X$ and $\lambda>a$, because $(\lambda-A)\int_0^\infty e^{-\lambda t}S(t)x\,dt=Cx=R(c\,;A)^n x$ by (1.4). Consequently, $\lambda\in\rho(A)$ and $\int_0^\infty \lambda^n e^{-\lambda t}U(t)x\,dt=R(\lambda\,;A)x$ for $x\in X$ and $\lambda>a$. It follows from [2, Theorem 3.1] that $U(t)$, $t\geq 0$, satisfy (1.6). So that $\{U(t)\,;t\geq 0\}$ is an n-times integrated semigroup and its generator is A.

Finally, to prove that (i) implies (iii) let A be the generator of an n-times integrated semigroup $\{U(t);\ t\geq 0\}$ and $\|U(t)\|\leq Ke^{\omega t}$ for $t\geq 0$. Then, by the definition of the generator, $(\omega,\infty)\subset\rho(A)$ and $R(\lambda\,;A)x=\int_0^\infty \lambda^n e^{-\lambda t}U(t)x\,dt$ for $x\in X$ and $\lambda>\omega$. So, $(\lambda-A)\int_0^\infty e^{-\lambda t}(\sum_{k=0}^{n-1}(t^k/k!)A^k x+U(t)A^n x)dt=(\lambda-A)\sum_{k=0}^{n-1}\lambda^{-(k+1)}A^k x+\lambda^{-n}A^n x=x$ for $x\in D(A^n)$ and $\lambda>|\omega|$ and hence $R(\lambda\,;A)x=\int_0^\infty e^{-\lambda t}(\sum_{k=0}^{n-1}(t^k/k!)A^k x+U(t)A^n x)\,dt$ for $x\in D(A^n)$ and $\lambda>|\omega|$. Differentiating $m-1$ times with respect to λ,

$$(-1)^{m-1}(m-1)!\,R(\lambda\,;A)^m x=\int_0^\infty(-t)^{m-1}e^{-\lambda t}\left(\textstyle\sum_{k=0}^{n-1}(t^k/k!)A^k x+U(t)A^n x\right)dt$$

for $x\in D(A^n)$ and $\lambda>|\omega|$. Hence

$$\begin{aligned}&(m-1)!\,\|R(\lambda\,;A)^m R(c\,;A)^n x\|\\ &\quad\leq\int_0^\infty t^{m-1}e^{-\lambda t}\left(\textstyle\sum_{k=0}^{n-1}(t^k/k!)\|A^k R(c\,;A)^n\|+Ke^{\omega t}\|A^n R(c\,;A)^n\|\right)\|x\|\,dt\\ &\quad\leq(m-1)!\,M\|x\|/(\lambda-a)^m\end{aligned}$$

for $x\in X$, $\lambda>a$ and $m\geq 1$, where $a=\max\{1,|\omega|\}$ and $M=2\max\{\|A^k R(c\,;A)^n\|,\ k=0,1,\cdots,n-1\,;\ K\|A^n R(c\,;A)^n\|\}$. Q.E.D.

Proof of Theorem 2. We may assume that $s(A)<0$ by replacing A by $A-w$. (See [1, Proof of Theorem 4.1].) By [1, Lemma 2.1]

(2.6) $R(0\,;A)=R(\lambda\,;A)+\lambda R(\lambda\,;A)^2+\cdots+\lambda^{m-1}R(\lambda\,;A)^m+\lambda^m R(\lambda\,;A)^m R(0\,;A)$

for $m\geq 1$ and $\lambda\geq 0$, and

$$\sup\{\|\lambda^m R(\lambda\,;A)^m R(0\,;A)\|\,;\ m\geq 1,\ \lambda\geq 0\}<+\infty.$$

By virtue of Theorem 1, A is the generator of a once integrated semigroup $\{U(t)\,;\,t\geq 0\}$ and the C-$c.i.g.$ of a C-semigroup $\{S(t)\,;\,t\geq 0\}$ with $C=R(0\,;\,A)$, and by (1.10) and (2.5) $U(t)x=-A\int_0^t S(s)x\,ds=R(0\,;\,A)x-S(t)x$ for $x\in E$ and $t\geq 0$. It follows from (2.6) that for $0\leq s\leq t$ and $\lambda>0$, $(\lambda R(\lambda\,;\,A))^{[\lambda s]}R(0\,;\,A)-(\lambda R(\lambda\,;\,A))^{[\lambda t]}R(0\,;\,A)=(\lambda R(\lambda\,;\,A))^{[\lambda s]}(R(0\,;\,A)-(\lambda R(\lambda\,;\,A))^{[\lambda t]-[\lambda s]}R(0\,;\,A))=(\lambda R(\lambda\,;\,A))^{[\lambda s]}\sum_{k=0}^{[\lambda t]-[\lambda s]-1}\lambda^k R(\lambda\,;\,A)^{k+1}$ if $[\lambda t]>[\lambda s]$, $=0$ if $[\lambda t]=[\lambda s]$, which yields $(\lambda R(\lambda\,;\,A))^{[\lambda s]}R(0\,;\,A)\geq(\lambda R(\lambda\,;\,A))^{[\lambda t]}R(0\,;\,A)$, where [ ] denotes the Gaussian bracket. Since $\lim_{\lambda\to\infty}(\lambda R(\lambda\,;\,A))^{[\lambda t]}R(0\,;\,A)x=S(t)x$ for $x\in E$ and $t\geq 0$ (see [6, Theorem 1.3]), we have that $S(s)\geq S(t)$ for $0\leq s\leq t$. Combining this with $U(t)=R(0\,;\,A)-S(t)$, we see that $0=U(0)\leq U(s)\leq U(t)$ for $0\leq s\leq t$.

Q.E.D.

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