35. On Automorphism Groups of Compact Riemann Surfaces of Genus 5

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Let X be a compact Riemann surface of genus $g \geqq 2$. A group AG of automorphism of X can be represented as a subgroup $R(X, AG)$ of $GL(g, C)$ as elements of AG operate in the g-dimensional module of abelian differentials on X. The purpose of this paper is to determine in case $g=5$ all subgroups of $GL(g, C)$ which are conjugate to some $R(X, AG)$ for some X and some AG. For the case $g=2$, 3, 4 the same problem was already solved: [3] for the case $g=2$; the result for $g=3$, 4 is not yet published. A more detailed account will be published elsewhere.

§0. **Preliminaries.** Let G be a finite subgroup of $GL(g, C)$ and let H be a non-trivial cyclic subgroup of G. Define two sets $CY(G)$ and $CY(G; H)$ as in [3]. If any element of $CY(G)$ is $GL(g, C)$-conjugate to a subgroup arising from Riemann surfaces of genus g, then we say that G stands the CY-test. Further we define $l(H; G)$ and $RH(G)$ as in [3]. If G stands the CY-test and $l(H; G)$ is a non-negative integer for every element H of $CY(G)$, then we say that G stands the RH-test. Let $RH(G)$ be $[g_0, n; e_1, \cdots, e_r]$ and let Γ be a Fuchsian group $\langle \alpha_1, \beta_1, \cdots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \cdots \gamma_r \rangle$ with relations $\prod_{j=1}^{r} \gamma_j \cdot \prod_{i=1}^{g_0} [\alpha_i, \beta_i]=1$, $\gamma_1^{e_1}=\cdots=\gamma_r^{e_r}=1$. If we have a surjective homomorphism $\phi: \Gamma \to G$ such that $\sharp\phi(\gamma_j)=e_j$ and $2-2\,\mathrm{Re}\,(\mathrm{tr}\,\phi(\gamma_j))>0$ $(1 \leqq j \leqq r)$, then we say that G stands the EX-test. If G stands the EX-test, then it can be shown that there exists an $R(X, AG)$ which is $GL(5, C)$-conjugate to G by taking a suitable ϕ [1, 2, 4].

Notations. We use following notations for economy of space.

$$A(a, b, c, d, e) = \begin{bmatrix} a &&&& \\ & b &&& \\ && c && \\ &&& d & \\ &&&& e \end{bmatrix}, \quad B(a, b, c, d, e) = \begin{bmatrix} a &&&& \\ & b &&& \\ && c && \\ &&& 0 & d \\ &&& e & 0 \end{bmatrix}, \quad C(a, b, c, d, e) = \begin{bmatrix} a &&&& \\ & 0 & b && \\ & c & 0 && \\ &&& d & 0 \\ &&& 0 & e \end{bmatrix},$$

$$D(a, b, c, d, e) = \begin{bmatrix} 0 & a &&& \\ b & 0 &&& \\ && c && \\ &&& d & \\ &&&& e \end{bmatrix}, \quad E(a, b, c, d, e) = \begin{bmatrix} a &&&& \\ & 0 & 0 & b & 0 \\ & 0 & c & 0 & 0 \\ & d & 0 & 0 & 0 \\ & 0 & 0 & 0 & e \end{bmatrix}, \quad F(a, b, c, d, e) = \begin{bmatrix} a &&&& \\ & 0 & 0 & 0 & b \\ & 0 & 0 & c & 0 \\ & 0 & d & 0 & 0 \\ & e & 0 & 0 & 0 \end{bmatrix},$$

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$$G(a,b,c,d,e)=\begin{bmatrix} a & & & & \\ & 0 & b & & \\ & c & 0 & & \\ & & & 0 & d \\ & & & e & 0 \end{bmatrix},\quad H(a,b,c,d,e)=\begin{bmatrix} a & & & & \\ & 0 & 0 & b & 0 \\ & 0 & 0 & 0 & c \\ & d & 0 & 0 & 0 \\ & 0 & e & 0 & 0 \end{bmatrix},\quad K(a,b,c,d,e)=\begin{bmatrix} 0 & a & & & \\ b & 0 & & & \\ & & 0 & 0 & c \\ & & 0 & d & 0 \\ & & e & 0 & 0 \end{bmatrix}$$

$$L(a,b,c,d,e)=\begin{bmatrix} a & & & & \\ & 0 & 0 & 0 & b \\ & c & 0 & 0 & 0 \\ & 0 & d & 0 & 0 \\ & 0 & 0 & e & 0 \end{bmatrix},\quad M(a,b,c,d,e)=\begin{bmatrix} a & & & & \\ & 0 & 0 & 0 & b \\ & 0 & 0 & c & 0 \\ & d & 0 & 0 & 0 \\ & 0 & e & 0 & 0 \end{bmatrix},\quad N(a,b,c,d,e)=\begin{bmatrix} a & & & & \\ & b & & & \\ & & 0 & 0 & c \\ & & d & 0 & 0 \\ & & 0 & e & 0 \end{bmatrix},$$

$$S=\begin{bmatrix} -i & & & & \\ & \zeta^5/\sqrt{2} & 0 & \zeta^7/\sqrt{2} & 0 \\ & 0 & 0 & 0 & i \\ & \zeta^7/\sqrt{2} & 0 & \zeta^5/\sqrt{2} & 0 \\ & 0 & i & 0 & 0 \end{bmatrix},\ \zeta=\zeta_8,\qquad U=\begin{bmatrix} -1 & & & & \\ & 0 & 0 & \zeta & 0 \\ & 0 & 0 & 0 & -1 \\ & \zeta^7 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 0 \end{bmatrix},\ \zeta=\zeta_8,$$

$$Q=\begin{bmatrix} i & & & & \\ & 0 & 0 & 1/\sqrt{2} & -i/\sqrt{2} \\ & 0 & 0 & i/\sqrt{2} & -1/\sqrt{2} \\ & -1/\sqrt{2} & -i/\sqrt{2} & 0 & 0 \\ & i/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix},$$

$$T=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & \zeta^5/\sqrt{2} & 0 & \zeta^5/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta^7/\sqrt{2} & 0 & \zeta^3/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},\ \zeta=\zeta_8,\qquad V=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathrm{W}=\begin{bmatrix} a & b & b & b & b \\ b & c & d & e & f \\ b & d & f & c & e \\ b & e & c & f & d \\ b & f & e & d & c \end{bmatrix}.\qquad \begin{array}{l} a=-1/5,\quad b=\sqrt{6}/5,\quad c=(3-\sqrt{5})/10, \\ d=-(1+\sqrt{5})/5,\quad e=-(1-\sqrt{5})/5, \\ f=(3+\sqrt{5})/10. \end{array}$$

Our study is to be made according to the following arrangements:

First, we consider $n=\sharp G$ which satisfies the Riemann-Hurwitz relation. Second, we consider cyclic groups which stands the CY-test. Third, that is the main part of this paper, we study non-cyclic groups of order $n=2^a\cdot 3^b\cdot 5^c\cdot 11^d$ with $0\leqq a\leqq 6$, $0\leqq b\leqq 1$, $0\leqq c\leqq 1$ and $0\leqq d\leqq 1$. All groups are solvable except for $n=60$ and 120 and so we can use some results of normalizers for the determination of groups. In the process, we consider groups which stand the CY-test. Next, we consider groups which stand the RH-test. Finally, we consider groups which stand the EX-test.

We shall give an outline of the method of determination in case $n=16$ in the following:

A. *Abelian groups.*

(I) If G is of type (8, 2), we have only $AG_1(16)$ by considering $N_{GL}(CG_2(8))$.

(II) If G is of type (4, 4), all considerable cases give contradiction.

(III) If G is of type (4, 2, 2), we have only $AG_2(16)$ considering

$N_{GL}(AG_2(8))$.

(IV) If G is of type (2, 2, 2, 2), we have only $AG_3(16)$ by considering $N_{GL}(AG_{10}(8))$.

B. *Non-abelian groups.* $G \triangleright H$ means H is a normal subgroup of G.

(a) If $G \triangleright CG_1(8)$, we have $G_1(16)$, $G_2(16)$ and $G_3(16)$.

(b) If $G \triangleright CG_3(8)$, we have $G_4(16)$.

(c) If $G \triangleright AG_1(8)$, we have $G_5(16)$, $G_6(16)$, $G_7(16)$, $G_8(16)$ and $G_9(16)$.

(d) If $G \triangleright AG_2(8)$, we have $G_{10}(16)$ and $G_{11}(16)$.

(e) If $G \triangleright AG_5(8)$, we have $G_{14}(16)$.

(f) If $G \triangleright AG_6(8)$, we have $G_{13}(16)$.

(g) If $G \triangleright AG_8(8)$, we have $G_{12}(16)$.

All the other cases give either contradiction or no new groups.

Remark. (1) In case $G \triangleright AG_9(8)$, we have a group which stands the CY-test but not the RH-test. (2) In case $G \triangleright CG_1(8)$, we have a group which stands the RH-test but not the EX-test.

Remark. $\mathcal{D} := \langle A(1, i, i, -i, -i), F(1, 1, 1, 1, 1)\rangle$ and $Q := \langle A(1, i, i, -i, -i), F(1, i, i, i, i)\rangle$ stand the RH-test but not the EX-test. In case $g=5$, we have no other such groups except $\mathcal{D}$, Q and some groups including $\mathcal{D}$ or Q.

§ 1. Cyclic groups.

1. $n=2$. (1) $G_1(2)=\langle A(1, 1, 1, -1, -1)\rangle$. (2) $G_2(2)=\langle 1, 1, -1, -1, -1)\rangle$. (3) $G_3(2)=\langle A(1, -1, -1, -1, -1)\rangle$. (4) $G_4(2)=\langle A(-1, -1, -1, -1, -1)\rangle$.

2. $n=3$. (1) $G_1(3)=\langle A(1, \omega, \omega, \omega^2, \omega^2)\rangle$. (2) $G_2(3)=\langle A(\omega, \omega, \omega, \omega^2, \omega^2)\rangle$.

3. $n=4$. (1) $CG_1(4)=\langle A(1, 1, -1, i, -i)\rangle$.
(2) $CG_2(4)=\langle A(1, i, i, -i, -i)\rangle$. (3) $CG_3(4)=\langle A(1, -1, i, i, -i)\rangle$.
(4) $CG_4(4)=\langle A(i, i, i, -i, -i)\rangle$. (5) $CG_5(4)=\langle A(-1, i, i, -i, -i)\rangle$.
(6) $CG_6(4)=\langle A(-1, i, i, i, -i)\rangle$.

4. $n=5$ $(\zeta=\zeta_5)$. (1) $G(5)=\langle A(1, \zeta, \zeta^2, \zeta^3, \zeta^4)\rangle$.

5. $n=6$ $(\zeta=\zeta_6)$. (1) $CG_1(6)=\langle A(1, \zeta, \zeta^4, \zeta^2, \zeta^5)\rangle$.
(2) $CG_2(6)=\langle A(-1, \zeta, \zeta, \zeta^5, \zeta^5)\rangle$. (3) $CG_3(6)=\langle A)\zeta, \zeta, \zeta^4, \zeta^2, \zeta^5)\rangle$.
(4) $CG_4(6)=\langle A(-1, \zeta, \zeta, \zeta^2, \zeta^5)\rangle$. (5) $CG_5(6)=\langle A(-1, \zeta, \zeta^4, \zeta^2, \zeta^5)\rangle$.

6. $n=8$ $(\zeta=\zeta_8)$. (1) $CG_1(8)=\langle A(1, \zeta, \zeta^5, \zeta^3, \zeta^7)\rangle$.
(2) $CG_2(8)=\langle A(\zeta^2, \zeta, \zeta, \zeta^5, \zeta^3)\rangle$. (3) $CG_3(8)=\langle A(\zeta^2, \zeta, \zeta^5, \zeta^3, \zeta^7)\rangle$.

7. $n=10$ $(\zeta=\zeta_{10})$. (1) $CG(10)=\langle A(-1, \zeta, \zeta^7, \zeta^3, \zeta^9)\rangle$.

8. $n=11$ $(\zeta=\zeta_{11})$. (1) $G_1(11)=\langle A(\zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6)\rangle$.
(2) $G_2(11)=\langle A(\zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5)\rangle$.

9. $n=12$ $(\zeta=\zeta_{12})$. (1) $CG(12)=\langle A(\zeta^3, \zeta, \zeta^{10}, \zeta^8, \zeta^5)\rangle$.

10. $n=15$ $(\zeta=\zeta_{15})$. (1) $CG(15)=\langle A(\zeta^{10}, \zeta, \zeta^2, \zeta^8, \zeta^4)\rangle$.

11. $n=20$ $(\zeta=\zeta_{20})$. (1) $CG(20)=\langle A(\zeta^{15}, \zeta, \zeta^7, \zeta^3, \zeta^9\rangle$.

12. $n=22$ $(\zeta=\zeta_{22})$. (1) $CG(22)=\langle A(\zeta, \zeta^{13}, \zeta^3, \zeta^{15}, \zeta^5)\rangle$.

§ 2. Non-cyclic groups G of order $2^k (2 \leq k \leq 6)$.

1. $n=4$. (1) $G_1(4)=\langle G_1(2), A(1, 1, -1, 1, -1)\rangle$.
(2) $G_2(4)=\langle G_1(2), G_3(2)\rangle$. (3) $G_3(4)=\langle G_1(2), A(1, -1, -1, -1, 1)\rangle$.
(4) $G_4(4)=\langle G_1(2), A(-1, -1, -1, 1, 1)\rangle$.

(5) $G_5(4)=\langle G_1(2), A(-1, -1, -1, 1, -1)\rangle$.
(6) $G_6(4)=\langle G_2(2), A(-1, -1, 1, 1, -1)\rangle$.

2. $n=8$. A. *Abelian groups.*

(1) $AG_1(8)=\langle CG_1(4), A(1, -1, -1, 1, 1)\rangle$.
(2) $AG_2(8)=\langle CG_2(4), A(1, 1, -1, 1, -1)\rangle$.
(3) $AG_3(8)=\langle CG_2(4), A(-1, 1, 1, 1, -1)\rangle$.
(4) $AG_4(8)=\langle CG_2(4), A(-1, 1, -1, 1, -1)\rangle$.
(5) $AG_5(8)=\langle CG_3(4), A(-1, -1, 1, 1, 1)\rangle$.
(6) $AG_6(8)=\langle CG_3(4), A(-1, -1, 1, -1, 1)\rangle$.
(7) $AG_7(8)=\langle CG_4(4), A(1, 1, -1, 1, -1)\rangle$.
(8) $AG_8(8)=\langle CG_5(4), A(1, 1, -1, 1, -1)\rangle$.
(9) $AG_9(8)=\langle CG_6(4), G_1(2)\rangle$.
(10) $AG_{10}(8)=\langle G_1(4), A(1, -1, 1, 1, -1)\rangle$.
(11) $AG_{11}(8)=\langle G_1(4), A(-1, -1, 1, 1, 1)\rangle$.
(12) $AG_{12}(8)=\langle G_1(4), A(-1, -1, 1, 1, -1)\rangle$.
(13) $AG_{13}(8)=\langle G_2(4), A(-1, 1, -1, 1, -1)\rangle$.

B. *Non-abelian groups.* (1) $G_1(8)=\langle CG_1(4), B(1, -1, 1, 1, 1)\rangle$. (2) $G_2(8)=\langle CG_1(4), B(-1, -1, 1, 1, 1)\rangle$. (3) $G_3(8)=\langle CG_2(4), F(-1, 1, 1, 1, 1)\rangle$. (4) $G_4(8)=\langle CG_5(4), F(1, 1, 1, 1, 1)\rangle$. (5) $G_5(8)=\langle CG_2(4), F(-1, i, i, i, i)\rangle$.

3. $n=16$. A. *Abelian groups.*

(1) $AG_1(16)=\langle CG_2(8), A(1, -1, 1, -1, 1)\rangle$.
(2) $AG_2(16)=\langle AG_2(8), A(-1, 1, 1, 1, -1)\rangle$.
(3) $AG_3(16)=\langle AG_{10}(8), A(-1, 1, 1, 1, -1)\rangle$.

B. *Non-abelian groups.*

(1) $G_1(16)=\langle CG_1(8), G(1, 1, 1, 1, 1)\rangle$. (2) $G_2(16)=\langle CG_1(8), F(-1, 1, 1, 1, 1)\rangle$.
(3) $G_3(16)=\langle CG_1(8), H(-1, 1, 1, 1, 1)\rangle$. (4) $G_4(16)=\langle CG(8), G(1, 1, 1, 1, 1)\rangle$.
(5) $G_5(16)=\langle AG_1(8), B(-1, 1, 1, 1, 1)\rangle$. (6) $G_6(16)=\langle AG_1(8), G(1, 1, 1, 1, 1)\rangle$.
(7) $G_7(16)=\langle AG_1(8), C(-1, 1, 1, 1, 1)\rangle$.
(8) $G_8(16)=\langle AG_1(8), C(-1, 1, 1, 1, -1)\rangle$.
(9) $G_9(16)=\langle AG_1(8), G(1, 1, 1, i, i)\rangle$. (10) $G_{10}(16)=\langle AG_2(8), G(-1, 1, 1, 1, 1)\rangle$.
(11) $G_{11}(16)=\langle AG_2(8), H(-1, 1, 1, 1, 1)\rangle$.
(12) $G_{12}(16)=\langle AG_8(8), H(1, 1, 1, 1, 1)\rangle$. (13) $G_{13}(16)=\langle AG_6(8), K(1, 1, 1, 1, 1)\rangle$.
(14) $G_{14}(16)=\langle AG_5(8), D(1, 1, 1, -1, 1)\rangle$.

4. $n=32$. (1) $G_1(32)=\langle AG_1(16), E(-1, 1, -1, 1, -1)\rangle$.
(2) $G_2(32)=\langle AG_2(16), E(-i, i, i, i, 1)\rangle$.
(3) $G_3(32)=\langle AG_2(16), H(-1, 1, 1, 1, 1)\rangle$.
(4) $G_4(32)=\langle AG_3(16), G(-1, 1, 1, 1, 1)\rangle$.
(5) $G_5(32)=\langle G_1(16), H(-1, 1, 1, 1, 1)\rangle$. (6) $G_6(32)=\langle G_{12}(16), L(i, 1, i, 1, -i)\rangle$.
(7) $G_7(32)=\langle G_4(16), Q\rangle$.

5. $n=64$. (1) $G_1(64)=\langle G_1(32), S\rangle$. (2) $G_2(64)=\langle G_4(32), M(i, 1, 1, 1, 1)\rangle$.

§3. Non-cyclic groups G of order $2^k\cdot 3$ $(1\leqq k\leqq 6)$.

1. $n=6$. (1) $G_1(6)=\langle G_1(3), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(6)=\langle G_2(3), F(-1, 1, 1, 1, 1)\rangle$.

2. $n=12$. A. *Abelian groups.*

(1) $AG_1(12)=\langle CG_1(6), (-1, 1, -1, -1, 1)\rangle$.
(2) $AG_1(12)=\langle CG_1(6), A(-1, -1, -1, -1, 1)\rangle$.

B. *Non-abelian groups.* (1) $G_1(12)=\langle CG_1(6), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(12)=\langle CG_1(6), F(-1, 1, 1, 1, 1)\rangle$. (3) $G_3(12)=\langle CG_2(6), F(1, 1, 1, 1, 1)\rangle$.
(4) $G_4(12)=\langle CG_5(6), F(1, 1, 1, 1, 1)\rangle$. (5) $G_5(12)=\langle CG_1(6), F(1, i, 1, 1, i)\rangle$.
(6) $G_6(12)=\langle CG_2(6), F(i, i, i, i, i)\rangle$. (7) $G_7(12)=\langle AG_1(6), N(\omega, \omega^2, 1, 1, 1)\rangle$.

3. $n=24$. A. *Abelian groups.*
(1) $AG_1(24)=\langle CG(12), A(1, 1, -1, -1, 1)\rangle$.

B. *Non-abelian groups.* (1) $G_1(24)=\langle AG_1(12), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(24)=\langle AG_1(12), F(i, i, i, i, i)\rangle$. (3) $G_3(24)=\langle AG_2(12), F(1, 1, 1, 1, 1)\rangle$.
(4) $G_4(24)=\langle G_7(12), A(1, -1, -1, 1, 1)\rangle$.
(5) $G_5(24)=\langle G_7(12), A(-1, -1, -1, -1, -1)\rangle$.
(6) $G_6(24)=\langle G_7(12), A(-1, -1, 1, -1, -1)\rangle$.
(7) $G_7(24)=\langle G_7(12), K(1, 1, 1, 1, -1)\rangle$.

4. $n=48$. (1) $G_1(48)=\langle AG_1(24), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(48)=\langle G_4(24), A(-1, 1, -1, 1, 1)$. (3) $G_3(48)=\langle G_5(24), K(i, i, i, i, i)\rangle$.
(4) $G_4(48)=\langle G_7(24), A(-1, -1, 1, 1, 1)\rangle$. (5) $G_5(48)=\langle G_6(24), K(i, i, 1, 1, 1)\rangle$.

5. $n=96$. (1) $G_1(96)=\langle G_2(48), K(1, 1, -1, -1, -1)\rangle$.
(2) $G_2(96)=\langle G_2(32), T\rangle$.

6. $n=192$. (1) $G(192)=\langle G_2(96), U\rangle$.

§4. Non-cyclic groups G of order $2^k\cdot 5$ $(1\leqq k\leqq 5)$.

1. $n=10$. (1) $G_1(10)=\langle G(5), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(10)=\langle G(5), F(-1, 1, 1, 1, 1)\rangle$.

2. $n=20$. (1) $G_1(20)=\langle CG(10), F(1, 1, 1, 1, 1)\rangle$.
(2) $G_2(20)=\langle CG(10), F(i, i, i, i, i)\rangle$.

3. $n=40$. (1) $G(40)=\langle CG(20), F(1, 1, 1, 1, 1)\rangle$.

4. $n=80$. (1) $G(80)=\langle AG_3(16), V\rangle$.

5. $n=160$. (1) $G(160)=\langle G(80), G(-1, 1, 1, 1, 1)\rangle$.

§5. Non-cyclic groups G of order $2^k\cdot 3\cdot 5$ $(1\leqq k\leqq 3)$.

1. $n=30$. (1) $G(30)=\langle CG(15), F(-1, 1, 1, 1, 1)\rangle$.

2. $n=60$. (1) $G(60)=\langle G(5), W\rangle$.

3. $n=120$. (1) $G(120)=\langle G(60), A(-1, -1, -1, -1, -1)\rangle$.

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