22. *Local Analytic Dimensions of a Subanalytic Set*

By Heisuke HIRONAKA, M. J. A.

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The class of subanalytic sets in a real-analytic manifold M is, by definition, generated by the images of proper real-analytic maps into M with respect to the elementary set-theoretical operations, i. e., finite union, finite intersection and difference. A subanalytic set X in M admits a locally finite stratification in which strata are locally closed real-analytic submanifolds (smooth and connected) of M, say X_i, and are subanalytic themselves in M. (cf. [3]) This enables us to define the topological dimension of X at each point x of M as follows:

$$t\text{-}\dim_x X = \max\{\dim X_i : x \in \bar{X}_i\}$$

which is independent of the choice of stratification.

This article is concerned with other kinds of local dimension of X. First of all, it is known that the closure $\bar{X}$ is also subanalytic in M and is in fact the image of a proper real-analytic map, say $f: Y \to M$. Here we assume that Y is a reduced real-analytic space because the reduction (killing the nilpotents in the structure sheaf of functions) does not affect the image set. Then, for each point $x \in \bar{X}$, we let

$$\boldsymbol{A}_x(X) = \{h \in \mathcal{O}_{M,x} : (h \circ f)_y = 0 \text{ for all } y \in f^{-1}(x)\}$$

$$\boldsymbol{F}_x(X) = \{\hat{h} \in \hat{\mathcal{O}}_{M,x} : (\hat{h} \circ \hat{f})_y = 0 \text{ for all } y \in f^{-1}(x)\}$$

where $\mathcal{O}_{M,x}$ denotes the ring of germs of analytic functions at x on M, $(\ )_y$ does the germ at y, $\hat{\mathcal{O}}_{M,x}$ does the formal completion of $\mathcal{O}_{M,x}$ and $(\ \circ \hat{f})_y$ does the completion map of $(\ \circ f)_y : \mathcal{O}_{M,x} \to \mathcal{O}_{Y,y}$.

Definition. The Krull dimension of $\mathcal{O}_{M,x}/\boldsymbol{A}_x(X)$ is called the analytic dimension of X at x, denoted by $a\text{-}\dim_x X$, while that of $\hat{\mathcal{O}}_{M,x}/\boldsymbol{F}_x(X)$ is called the formal dimension of X at x, denoted by $f\text{-}\dim_x X$.

The dimensions, analytic and formal, defined as above are in fact independent of the choice of f and depend only on the image $\bar{X}$. So are the ideals $\boldsymbol{A}_x(X)$ and $\boldsymbol{F}_x(X)$. Obviously $t\text{-}\dim_x X \le f\text{-}\dim_x X \le a\text{-}\dim_x X$ and the strict inequalities are possible. (cf. [2] and [3])

The result of this article is

Theorem. *Let X be any subanalytic subset of M. Then there exists a locally finite stratification, say $X = \bigcup_i X_i$, with strata X_i all subanalytic in M, having the following properties: For a sufficiently small open neighborhood U_i of X_i in M for each i, we have*

1) *X_i is a closed real-analytic submanifold of U_i*

2) *there exists a coherent ideal sheaf $\boldsymbol{A}_i$ in $\mathcal{O}_{U_i}$ having stalks $\boldsymbol{A}_{ix} = \boldsymbol{A}_x(X)$ for all $x \in X_i$*

3) *$\hat{U}_i$ denoting the formal completion of U_i with respect to the powers*

of the ideal sheaf of X_i, *there exists a coherent ideal sheaf* F_i *in* $\mathcal{O}_{\hat{U}_i}$ *such that* $F_x(X)=F_{ix}\hat{\mathcal{O}}_{\hat{U}_i,x}$ *for all* $x \in X_i$.

Corollary. *For every integer* $d \geq 0$, *the set of points* x *of* M *with* $a\text{-dim}_x(x)=d$ *is a subanalytic set in* M. *Similarly, the set of* x *with* $f\text{-dim}_x(X)=d$ *is a subanalytic set in* M. (cf. conjectures in [1])

Here we describe a rough sketch of the proof of the theorem, whose details will be published elsewhere.

Step 1. There exists a locally finite stratification of the closure of X, say $\bar{X}=\bigcup X_i$, having the following properties:

a) X_i is a locally closed real-analytic submanifold in M, connected and subanalytic in M.

b) $\bar{X}$ is the image of a proper real-analytic map $f: Y \to M$ such that Y is a reduced real-analytic space (may even be assumed to be smooth by resolution of singularities) and such that $\oplus_{m=0}^{\infty} H_i^m/H_i^{m+1}$ is flat as $\mathcal{O}_{X_i}$-module, where H_i is the ideal sheaf in $\mathcal{O}_Y$ within a small neighborhood of $f^{-1}(X_i)$ which is generated by the ideal of X_i in M with reference to f.

Step 2. Let U_i be a sufficiently small neighborhood of X_i in M, in which X_i is closed. Let $\hat{U}_i$ be the completion of U_i by the powers of the ideal sheaf of X_i, and let $\hat{Y}_i$ be the completions of $Y \mid f^{-1}(U_i)$ by the powers of the H_i in Step 1. For each point x of X_i, let J_x be the intersection of the kernels of the homomorphisms $\mathcal{O}_{\hat{U}_i,x} \to \mathcal{O}_{\hat{Y}_i,y}$ for all $y \in f^{-1}(x)$. Then there exists a coherent ideal sheaf J in $\mathcal{O}_{\hat{U}_i}$ whose stalks are those J_x.

Step 3. The last step is a lemma in complex-analytic geometry.

Lemma. *Let* Z *be a closed complex-analytic subspace of a complex-analytic manifold* V. *Let* $\hat{V}$ *be the completion of* V *by the powers of the ideal sheaf of* Z *in* $\mathcal{O}_V$. *Let* $\hat{G}$ *be any coherent ideal sheaf in* $\mathcal{O}_{\hat{V}}$. *Then there exists a coherent ideal sheaf* G' *in* $\mathcal{O}_V$ *within a small neighborhood of each stratum* Z_j *of a locally finite complex-analytic stratification of* Z *such that for every point* z *of* Z_j *the stalk* G'_z *is the kernel of the natural homomorphism* $\mathcal{O}_{V,z} \to \mathcal{O}_{\hat{V},z}/\hat{G}_z$.

Using the resolution of singularities, the proof of the lemma is reduced to the following fact: Let $x=(x_1, x_2, \cdots, x_n)$ and y be complex variables. Let $F(x, y)$ be a formal power series in the variable y in which the coefficients are holomorphic functions in a fixed neighborhood of 0 in the space of x. If $F(x, y)$ is divergent at $(0, 0)$ then so is it at every $(x, 0)$ with x sufficiently close to 0.

Remark. In the above lemma, if V is a complexification of a real-analytic manifold and if the ideal sheaf of Z and $\hat{G}$ are generated by real-analytic and real-formal functions, then G' is also generated by real-analytic functions. In the application to our situation, we let Z, V be complexifications of X_i, U_i. The flatness in Step 1 plays a very important role in proving the statement of Step 2. Moreover it is needed in proving that J_x of Step 2 generates $F_x(X)$ in the completion of $\mathcal{O}_{U_i,x}$.

References

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[2] Gabrielov, A. M.: Formal relations between analytic functions. Math. U.S.S.R. Izvestija, **7**, 1056–1088 (1973).

[3] Hironaka, Heisuke: Subanalytic sets. Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki. Kinokuniya, Tokyo, pp. 453–493 (1973).