2. Aitken-Steffensen Acceleration and a New Addition Formula for Fibonacci Numbers[†]

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Interesting addition formulae for Fibonacci numbers and for $\cot(x)$ are discovered by the third author through various numerical observations. It is verified rigorously for the first time by the first author by mathematical induction and then is proved more directly by the use of detailed properties of Cauchy matrix by the second author [3]. We will discuss these formulae in details elsewhere ([1] and [2]).

1. Addition formula. Consider the three parameter family of functions:

(1.1) $$p(x)=p(\alpha, \beta, \gamma ; x)=(\alpha\gamma^x+\beta)/(\gamma^x-1),$$

with $\alpha+\beta\neq0$, $\gamma\neq0$. For $\alpha=(\sqrt{5}-1)/2$,

(1.2) $$p(n)=p(\alpha, \alpha^{-1}, -\alpha^{-2} ; n)=F_{n-1}/F_n, \qquad n=1, 2, 3, \cdots$$

is the ratio of the consecutive Fibonacci numbers F_{n-1} and F_n:

(1.3) $$F_0=0, \quad F_1=1, \quad F_{n+1}=F_n+F_{n-1}, \quad n=1, 2, 3, \cdots.$$

The other important examples are

(1.4) $$\coth(x)=p(1, 1, e^2 ; x)$$

and

$$\cot(x)=p(\sqrt{-1}, \sqrt{-1}, \exp(2\sqrt{-1}) ; x).$$

We state the main theorem:

Theorem 1.1. *For $m=2, 3, 4, \cdots$ and for any set $(x_1, \cdots, x_m, y_1, \cdots, y_m)$ of arbitrary complex numbers the function $p(x)$ given by (1.1) satisfies:*

(1.5) $$p(x_1+\cdots+x_m+y_1+\cdots+y_m)=1/{}^t1_m(p(x_i+y_j))^{-1}1_m$$

$$=-\det(p(x_i+y_j))/\det\begin{vmatrix} p(x_i+y_j) & 1_m \\ {}^t1_m & 0 \end{vmatrix},$$

provided that $\gamma^{x_i+y_j}\neq1$ for $1\leq i, j\leq m$. Here $(p(x_i+y_j))$ denotes an $m\times m$ matrix with the indicated (i, j) components, 1_m is the column m-vector with all components 1, ${}^t1_m$ is its transposition.

When applying this formula to the sequences (1.2) we consider the case when x_i and y_j are natural numbers.

By (1.4) we obtain

(1.6) $$\cot(x_1+\cdots+x_m+y_1+\cdots+y_m)=1/{}^t1_m(\cot(x_i+y_j))^{-1}1_m.$$

In particular, putting $x_i=x-iy$ and $y_j=jy$, we have

(1.7) $\cot{(mx)} = 1/{}^t1_m(\cot{(x+(j-i)y)})^{-1}1_m.$

2. **Aitken-Steffensen acceleration.** It is well known [4] that any linearly convergent sequence p_n may be transformed into a better convergent one $(A_2p)_n$ by the Aitken-Steffensen transformation

(2.1) $(A_2p)_n = (p_{n+1}p_{n-1}-p_n^2)/(p_{n+1}-2p_n+p_{n-1})$

$$= -\begin{vmatrix} p_n & p_{n+1} \\ p_{n-1} & p_n \end{vmatrix} \Big/ \begin{vmatrix} p_n & p_{n+1} & 1 \\ p_{n-1} & p_n & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 1/(1,1)\begin{bmatrix} p_n & p_{n+1} \\ p_{n-1} & p_n \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The following extended "Aitken-Steffensen transformations", known as the "ε-algorithm" ([4]),

(2.2) $(A_mp)_n = 1/{}^t1_m(p_{n+j-i})^{-1}1_m,$ $m=2,3,4,\cdots$

are also frequently used to accelerate the convergence of sequences. Considering (1.5) in the case $x_i=n-i,$ and $y_j=j,$ we have:

Corollary to Theorem 1.1. *For* $m=2,3,4,\cdots$ *the sequence* $p_n=p(n)$ *given by* (1.1) *satisfies*

(2.3) $(A_mp)_n = p_{mn}.$

The present work begins by the discovery of this fact concerning $p_n = p(n)$ of (1.2) through numerical observation for $m=2.$ Observe the sequence (1.2); we obtain in fact:

(2.4)

n	1	2	3	4	5	6	7
p_n	0	1	1/2	2/3	3/5	5/8	8/13
$(A_2p)_n$		2/3	5/8	13/21	34/55	89/144	233/377.

Note that p_n converges to $(\sqrt{5}-1)/2$ as n tends to infinity. The above numerical data show that p_n is transformed into the new sequence $(A_2p)_n = p_{2n}$ which converges to the same limit doubly fast. It is natural to conjecture that $(A_3p)_n = p_{3n}, (A_4p)_n = p_{4n}, \cdots$ and so on. These relations can be easily confirmed numerically and it is easy to verify $(A_2p)_n = p_{2n}$ and $(A_3p)_n = p_{3n}$ directly. But to establish it in general case we need some results of pure mathematics about matrices and determinants. The first rigorous verification of Theorem 1.1 is done by mathematical induction with respect to m, by the use of Laplace expansion theorem for determinants.

3. **Cauchy matrix.** For a set $(a_1, \cdots, a_m, b_1, \cdots, b_m)$ of mutually distinct arbitrary complex numbers we consider the $m \times m$ Cauchy matrix and its modification

(3.1) $C = C(a,b) = \left(\dfrac{1}{a_i-b_j}\right),$

(3.2) $A = A(\alpha,\beta,a,b) = \left(\dfrac{\alpha a_i+\beta b_j}{a_i-b_j}\right).$

α, β being complex constants such that $\alpha+\beta \neq 0.$ It is well known (Cauchy's lemma [5]) that:

(3.3) $\det{C(a,b)} = \prod_{i<j}(a_i-a_j)(b_j-b_i)/\prod_{i,j}(a_i-b_j).$

Using this we can show that

(3.4) $\det A(\alpha, \beta, a, b) = (\alpha + \beta)^{m-1}(\alpha\sigma_m(a) + \beta\sigma_m(b)) \det C(a, b)$

where $\sigma_m(a) = a_1 \cdots a_m$ and $\sigma_m(b) = b_1 \cdots b_m$. Investigating these matrices in details we can show :

　　Theorem 3.1. *For the $m \times m$ matrices A and C we have*

(3.5) $1/{}^t 1_m A^{-1} 1_m = (\alpha\sigma_m(a) + \beta\sigma_m(b))/(\sigma_m(a) - \sigma_m(b))$

and

(3.6) $1/{}^t 1_m C(a, b)^{-1} 1_m = 1/(a_1 + \cdots + a_m - b_1 - \cdots - b_m)$.

　　In the case $a_i = \gamma^{x_i}$ and $b_j = \gamma^{-y_j}$, we have (1.5) from (3.5).

References

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