115. *Casson's Invariant for Homology 3-Spheres and the Mapping Class Group*

By Shigeyuki MORITA
Department of Mathematics, Tokyo Institute of Technology

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1. Introduction. Let Σ_g be a closed orientable surface of genus $g \geqq 2$ and let $\mathcal{M}_g$ be its mapping class group. Let $\mathcal{I}_g$ be the subgroup of $\mathcal{M}_g$ consisting of all mapping classes which act on the homology of Σ_g trivially. It is called the Torelli group of genus g and we have a short exact sequence $1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \boldsymbol{Z}) \to 1$ where $Sp(2g, \boldsymbol{Z})$ is the Siegel modular group. Recently Johnson has obtained several fundamental results concerning the structure of the Torelli group. Among other things he enumerated the Birman-Craggs homomorphisms $\mathcal{I}_g \to \boldsymbol{Z}/2$, which are defined by using the Rohlin invariant of homology 3-spheres (see [1]), and investigated the relationship between them and another abelian quotient of $\mathcal{I}_g$ which he constructed by making use of the action of $\mathcal{I}_g$ on a certain nilpotent quotient of $\pi_1(\Sigma_g)$ (see [3] [4]).

Now the purpose of the present note is to announce our recent related results. Roughly speaking, we have first "lifted" Johnson's result mentioned above in terms of Casson's invariant for homology 3-spheres ([2]) rather than the Rohlin invariant, and then put the computations forward by one step. As a result we can prove the existence of a new method of defining Casson's invariant (see Theorem 9).

2. Johnson's method. Here we briefly recall Johnson's method of investigating the mapping class groups (see [5] for details). For simplicity here we only consider the mapping class group $\mathcal{M}_{g,1}$ of Σ_g relative to an embedded disc $D^2 \subset \Sigma_g$. Write Γ_1 for $\pi_1(\Sigma_g^0)$ ($\Sigma_g^0 = \Sigma_g \setminus \mathring{D}^2$) and inductively define $\Gamma_{k+1} = [\Gamma_k, \Gamma_1]$ ($k=1, 2, \cdots$). We may call $N_k = \Gamma_1/\Gamma_k$ the k-th nilpotent quotient of Γ_1. We simply write H for $N_2 = H_1(\Sigma_g, \boldsymbol{Z})$ and choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of it. Let $\mathcal{L}$ be the free graded Lie algebra on x_i, y_j and let $\mathcal{L}_k$ be the submodule of $\mathcal{L}$ consisting of all homogeneous elements of degree k. It is a classical result that there exists a natural isomorphism $\Gamma_k/\Gamma_{k+1} \cong \mathcal{L}_k$ (see [7]) and we have a central extension $0 \to \mathcal{L}_k \to N_{k+1} \to N_k \to 1$. We have also natural isomorphisms $\mathcal{L}_2 \cong \wedge^2 H$ and $\mathcal{L}_3 \cong \wedge^2 H \otimes H / \wedge^3 H$. Now $\mathcal{M} = \mathcal{M}_{g,1}$ naturally acts on N_k and set $\mathcal{M}(k)$ to be the subgroup of $\mathcal{M}$ consisting of all elements which act on N_k trivially. $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$ and according to [6], $\mathcal{M}(3)$ is the subgroup of $\mathcal{M}$ generated by all Dehn twists on bounding simple closed curves on Σ_g^0. Hereafter we write $\mathcal{K}_{g,1}$ for $\mathcal{M}(3)$. Johnson constructed a homomorphism $\tau_k : \mathcal{M}(k) \to \mathcal{L}_k \otimes H$ such that $\operatorname{Ker} \tau_k = \mathcal{M}(k+1)$ and proved that $\operatorname{Im}\tau_2=\bigwedge^3 H\subset\mathcal{L}_2\otimes H$ (see [4] [5]). We have determined $\operatorname{Im}\tau_3$ (see Theorem 1 below).

3. Casson's invariant. Recently Casson [2] defined a new integral invariant λ for oriented homology 3-spheres whose $Z/2$ reduction is the Rohlin invariant. It can be characterized by the following property. Namely for any knot K in an oriented homology 3-sphere W, set $\lambda'(K)=(1/2)\Delta_K''(1)$ where $\Delta_K(t)=a_0+a_1(t+t^{-1})+\cdots$ is the Alexander polynomial of K with $\Delta_K(1)=1$. Then $\lambda(W')=\lambda(W)+\lambda'(K)$, where W' is the homology sphere obtained from W by performing $1/1$ Dehn surgery on K.

4. Statement of the main results. Consider the basis $x_i\wedge y_j$ $(i, j=1, \cdots, g)$, $x_i\wedge x_j$ $(i<j)$, $y_i\wedge y_j$ $(i<j)$ of $\bigwedge^2 H$ and write t_i $\left(i=1, \cdots, \binom{2g}{2}\right)$ for these elements (in any order). Let T be the submodule of $\bigwedge^2 H\otimes\bigwedge^2 H\subset\bigwedge^2 H\otimes H^2$ generated by $t_i\otimes t_i$ and $t_i\otimes t_j+t_j\otimes t_i$ $(i\neq j)$. Hereafter we simply write $t_i\leftrightarrow t_j$ for $t_i\otimes t_j+t_j\otimes t_i$. Let $\bar{T}$ be the image in $\mathcal{L}_3\otimes H$ of T under the projection $\bigwedge^2 H\otimes H^2\to(\bigwedge^2 H\otimes H/\bigwedge^3 H)\otimes H=\mathcal{L}_3\otimes H$. We can define a certain submodule $\bar{T}'$ of $\bar{T}$ of index a power of 2 (the precise definition of $\bar{T}'$ is omitted here).

Theorem 1. *The image of Johnson's homomorphism $\tau_3:\mathcal{K}_{g,1}\to\mathcal{L}_3\otimes H$ is $\bar{T}'$ which is a free abelian group of rank $1/3\ g^2(4g^2-1)$.*

Next let $\mathcal{F}$ be the set of all closed oriented surfaces F in S^3 of genus$\geqq g$ and let $\mathcal{E}mb$ be the set of all orientation preserving embeddings $f:\Sigma_g^0\to F\subset S^3$ with $F\in\mathcal{F}$. For each element $f\in\mathcal{E}mb$, we can define a map

$$\lambda(f):\mathcal{I}_{g,1}\longrightarrow Z$$

by setting $\lambda(f)(\phi)=\lambda(W_\phi)$ $(\phi\in\mathcal{I}_{g,1})$, where W_ϕ is the oriented homology 3-sphere obtained by cutting S^3 along F and then regluing the resulting two pieces by the map ϕ (cf. [1] [3]).

Lemma 2. *Although the map $\lambda(f)$ is not a homomorphism, we have $\lambda(f)(\phi_1\phi_2\phi_3)=\lambda(f)(\phi_1)+\lambda(f)(\phi_2)+\lambda(f)(\phi_3)$ if ϕ_1 and ϕ_3 are contained in $\mathcal{K}_{g,1}$. In particular the restriction of $\lambda(f)$ to $\mathcal{K}_{g,1}$ is a homomorphism.*

We define a homomorphism $\Lambda:\mathcal{K}_{g,1}\to Map(\mathcal{E}mb, Z)$ by setting $\Lambda(\phi)(f)=\lambda(f)(\phi)$ $(\phi\in\mathcal{K}_{g,1},\ f\in\mathcal{E}mb)$.

Now let K be a knot in an oriented homology 3-sphere W and let S be an oriented Seifert surface of it. Choose a symplectic basis $u_1, u_2, \cdots, u_{2h}$ of $H_1(S, Z)$ such that $u_i\cdot u_j=\delta_{i+h\ j}$ $(i<j)$ and let $U=(l(i, j))$, $l(i, j)=lk(u_i, u_j^+)$ be the associated Seifert matrix.

Proposition 3. *We have the equality*

$$\begin{aligned}\lambda'(K)=&\sum_{i=1}^{h}\{l(i, i)l(i+h, i+h)-l(i, i+h)l(i+h, i)\}\\&+2\sum_{i<j\leqq h}\{l(i, j)l(i+h, j+h)-l(i, j+h)l(j, i+h)\}.\end{aligned}$$

We define a commutative algebra $\mathcal{A}$ with unit 1 as follows. $\mathcal{A}$ has a generator $\overline{(u, v)}$ for any two elements $u, v\in H$ and we require the following relations to hold in $\mathcal{A}$:

(i) $\overline{(v, u)}=\overline{(u, v)}+u\cdot v$

(ii) $\overline{(n_1u_1+n_2u_2, v)}=n_1\overline{(u_1, v)}+n_2\overline{(u_2, v)}$ $(n_1, n_2 \in \boldsymbol{Z})$.

It is easy to see that $\mathcal{A}$ is nothing but the polynomial algebra over $\boldsymbol{Z}$ generated by the elements $\bar{x}_i$, $\bar{y}_i$, $\overline{(x_i, x_j)}$ $(i<j)$, $\overline{(y_i, y_j)}$ $(i<j)$ and $\overline{(x_i, y_j)}$ (we simply write $\bar{u}$ for $\overline{(u, u)}$). The group $Sp(2g, \boldsymbol{Z})$ acts naturally on $\mathcal{A}$.

For each bounding simple closed curve ω on Σ_g^0, let $\phi_\omega \in \mathcal{K}_{g,1}$ be the Dehn twist on ω and choose a symplectic basis $u_1, \cdots, u_{2h}$ of the homology of the subsurface of Σ_g^0 which ω bounds.

Theorem 4. *The correspondence*

$$\mathcal{K}_{g,1} \ni \phi_\omega \longmapsto \sum_{i=1}^{h} \{\bar{u}_i\bar{u}_{i+h}-\overline{(u_i, u_{i+h})}\,\overline{(u_{i+h}, u_i)}\} +2\sum_{i<j\leqq h} \{\overline{(u_i, u_j)}\,\overline{(u_{i+h}, u_{j+h})}-\overline{(u_i, u_{j+h})}\,\overline{(u_j, u_{i+h})}\} \in \mathcal{A}$$

for each generator $\phi_\omega \in \mathcal{K}_{g,1}$ defines a well-defined homomorphism $\rho: \mathcal{K}_{g,1} \to \mathcal{A}$ and the following diagram is commutative

$$\begin{tikzcd}
\mathcal{K}_{g,1} \arrow[r, "\rho"] \arrow[dr, "\Lambda"'] & \mathcal{A} \arrow[d, "\varepsilon"] \\
& Map(\mathcal{E}mb, \boldsymbol{Z})
\end{tikzcd}$$

where ε is the natural evaluation map.

Thus $\operatorname{Im}\rho$, which is contained in $\mathcal{A}_2$ (the submodule of $\mathcal{A}$ consisting of all elements of degree $\leqq 2$), can be considered as the space of all homomorphisms $\mathcal{K}_{g,1} \to \boldsymbol{Z}$ defined by the Casson invariant.

Proposition 5. *There exists a uniquely defined $Sp(2g, \boldsymbol{Z})$-equivariant homomorphism $\theta: T \to \mathcal{A}_2$ such that for any symplectic basis $u_1, \cdots, u_{2h}$ of the homology of any subsurface of Σ_g^0, the following equality holds:*

$$\begin{aligned}
&\theta((u_1\wedge u_{h+1}+\cdots+u_h\wedge u_{2h})\otimes(u_1\wedge u_{h+1}+\cdots+u_h\wedge u_{2h})) \\
&=\sum_{i=1}^{h} \{\bar{u}_i\bar{u}_{i+h}-\overline{(u_i, u_{i+h})}\,\overline{(u_{i+h}, u_i)}\} \\
&\quad +2\sum_{i<j\leqq h} \{\overline{(u_i, u_j)}\,\overline{(u_{i+h}, u_{j+h})}-\overline{(u_i, u_{j+h})}\,\overline{(u_j, u_{i+h})}\}.
\end{aligned}$$

The intersection $T \cap \wedge^3 H \otimes H$ is generated by the $Sp(2g, \boldsymbol{Z})$-orbits of the following three elements:

$$\begin{aligned}
s_1 &= x_1\wedge y_1 \longleftrightarrow x_2\wedge y_2 + y_1\wedge x_2 \longleftrightarrow x_1\wedge y_2 + x_2\wedge x_1 \longleftrightarrow y_1\wedge y_2 \\
s_2 &= x_1\wedge y_1 \longleftrightarrow x_2\wedge x_3 + y_1\wedge x_2 \longleftrightarrow x_1\wedge x_3 + x_2\wedge y_1 \longleftrightarrow y_1\wedge x_3 \\
s_3 &= x_1\wedge x_2 \longleftrightarrow x_3\wedge x_4 + x_2\wedge x_3 \longleftrightarrow x_1\wedge x_4 + x_3\wedge x_1 \longleftrightarrow x_2\wedge x_4
\end{aligned}$$

and we have $\theta(s_1)=-1$, $\theta(s_2)=\theta(s_3)=0$. Hence θ induces a homomorphism $\bar{\theta}: \bar{T} \to \mathcal{A}_2/\mathcal{A}_0$ where $\mathcal{A}_0$ is the submodule of $\mathcal{A}$ consisting of constants.

Theorem 6. *The following diagram is commutative*

$$\begin{tikzcd}
\dot{\mathcal{K}}_{g,1} \arrow[r, "\tau_3"] \arrow[dr, "\rho"'] & \bar{T}' \subset \bar{T} \arrow[d, "\bar{\theta}"] \\
& \mathcal{A}_2 \arrow[r] & \mathcal{A}_2/\mathcal{A}_0
\end{tikzcd}$$

where $\mathcal{A}_2 \to \mathcal{A}_2/\mathcal{A}_0$ is the projection.

Now the elements s_1, s_2 and s_3 vanish in $\mathcal{L}_3 \otimes H$ because of the "Jacobi identity"

$$[[\alpha, \beta], \gamma][[\beta, \gamma], \alpha][[\gamma, \alpha], \beta] = 1 \bmod \Gamma_4$$

which holds for any $\alpha, \beta, \gamma \in \Gamma_1$ (cf. [7]). However the above congruence does not hold mod Γ_5 and we can do the following. Define $E = \Gamma_3/\Gamma_5$ which is an abelian group. We can extend Johnson's method (see § 2) to obtain a homomorphism

$$\tilde{\tau}_3 : \mathcal{K}_{g,1} \longrightarrow \mathcal{H}om\,(N_3, E)$$

where $\mathcal{H}om\,(N_3, E)$ denotes the *abelian* group of all *crossed-homomorphisms* $N_3 \to E$, where N_3 acts on E naturally.

Theorem 7. *There exists a homomorphism* $\mathrm{Im}\,\tilde{\tau}_3 \to T$ *such that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{K}_{g,1} \arrow[r, "\tilde{\tau}_3"] \arrow[drr, "\rho"'] & \mathrm{Im}\,\tilde{\tau}_3 \arrow[r] & T \arrow[d, "\theta"] \\ & & \mathcal{A}_2 \end{tikzcd} .$$

Corollary 8. *The value of the map* $\lambda(f) : \mathcal{I}_{g,1} \to \boldsymbol{Z}$ *at any element* $\phi \in \mathcal{I}_{g,1}$ *depends only on the residue class of* ϕ *modulo the normal subgroup* $\mathcal{M}(5)$ *of* $\mathcal{I}_{g,1}$.

Theorem 9. *It is possible to define the Casson invariant of homology* 3-*spheres in terms of the action of the pasting maps of Heegaard decompositions on the fifth nilpotent quotient of the fundamental group of Heegaard surfaces.*

References

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