

12. Automorphism Groups of Real Algebraic Curves of Genus 3

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In [3] we have obtained the full list of automorphism groups of real (irreducible) algebraic hyperelliptic curves of genus 3. Let C be such a curve defined on R and $C(R)$ its real part. The automorphism group of $C(R)$, i.e., the group of its birational transformations, is one of the following: C_2 , $C_2 \times C_2$, $C_2 \times C_2 \times C_2$, D_6 and $C_2 \times D_4$, with the notation of [5]. Even more, if we denote by k the number of connected components of a non-singular model of $C(R)$, the following table recollects the automorphism groups according to the topological features of the curves:

Table I

$C \setminus C(R)$	k	C_2	$C_2 \times C_2$	$C_2 \times C_2 \times C_2$	D_6	$C_2 \times D_4$
Non-connected	4	*	*	*		*
Non-connected	2	*	*	*	*	*
Connected	1	*	*			
Connected	2	*	*	*		
Connected	3	*	*		*	

The sign * means the actual occurrence while no sign does non-occurrence.

In this note we extend this result to arbitrary real algebraic curves of genus 3. The technique is based upon the well-known functorial equivalence between real algebraic curves and bordered compact Klein surfaces (see [1]): Given a curve C of genus g , Alling and Greenleaf endow it with a structure of Klein surface, that is, a compact surface $X(C) = \{V \mid V \text{ is a valuation ring of } R(C), R \subset V\}$. Its boundary, consisting of those residually real valuation rings, is the non-singular model of $C(R)$.

Now, a Klein surface X may be expressed as D/Γ , where D is the hyperbolic plane and Γ is a non-Euclidean crystallographic (NEC) group, i.e., a discrete subgroup of isometries of the hyperbolic plane with compact quotient. (See [6].)

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Furthermore, a group of automorphisms of D/Γ , can be written as a quotient Γ'/Γ , Γ' being another NEC group. If $X(C)$ is D/Γ , Γ'/Γ is also a group of automorphisms of $C(\mathbf{R})$. This reduces the problem to a combinatorial question, since an NEC group Γ has a signature of the form

$$(g, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}),$$

where the m_i are called proper periods and the brackets $(n_{i1}, \dots, n_{is_i})$ are the period-cycles. The signature determines a presentation of the group in terms of generators and relations [6].

In order to obtain the group of automorphisms of a real algebraic curves of genus 3, we start determining the topological type of the Klein surface associated with the curve. Here we sketch the proof in one of the cases, and then we state the complete result. The detailed proofs will be found in [2].

Theorem. *Let C be an algebraic curve of genus 3 defined over the reals, which has a non-singular model with 3 connected components and such that $C \setminus C(\mathbf{R})$ is connected. Then the group of automorphisms of C is $C_2, C_2 \times C_2, D_6, D_3, D_4$ or S_4 .*

Proof. From Table I, we already know that $C_2, C_2 \times C_2$ and D_6 appear. Let $X = D/\Gamma$ be the Klein surface associated to C . Then the signature of Γ is $(1, -, [-], \{(-)(-)(-)\})$ by [8].

If we call G the group of automorphisms of D/Γ ([3]), its elements have order 2, 3, 4 or 6. Since the order of G is at most 24 by [7], G may only be one of the following groups :

$$C_4, C_6, D_3, C_2 \times C_4, C_2 \times C_2 \times C_2, D_4, Q, C_2 \times C_6, A_4, \\ C_2 \times C_2 \times C_2 \times C_2, C_2 \times D_4, C_2 \times D_6, C_2 \times A_4, S_4, C_2 \times \langle 2, 2, 3 \rangle.$$

Let us suppose $G = \Gamma'/\Gamma$. If the order of G is 24, then signature of Γ' is $(0, +, [-], \{(2, 2, 2, 3)\})$, and this group has a presentation $\langle c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \mid c_{10}^2 = c_{11}^2 = c_{12}^2 = c_{13}^2 = c_{14}^2 = (c_{10}c_{11})^2 = (c_{11}c_{12})^2 = (c_{12}c_{13})^2 = (c_{13}c_{14})^3 = c_{14}c_{10} = 1 \rangle$. Then an epimorphism θ from Γ' onto G must be given by

$$\theta(c_{10}) = x, \quad \theta(c_{11}) = 1, \quad \theta(c_{12}) = y, \quad \theta(c_{13}) = z, \quad \theta(c_{14}) = x,$$

verifying $x^2 = y^2 = z^2 = (xy)^4 = (yz)^2 = (zx)^3 = 1$, in order for its kernel to be Γ . Thus $G = S_4$.

Suppose now that G had order 16. Then Γ' would have signature $(0, +, [-], \{(2, 2, 2, 4)\})$ and then the epimorphism from Γ' onto G would verify that the number of period-cycles of its kernel were a divisor of 16. Since it must be 3, this is impossible. Similarly G may not have order 12.

Now we study the case of order 8. Then Γ' must have signature $(0, +, [-], \{(2, 2, 2, 2, 2)\})$ and so a presentation $\langle c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15} \mid c_{14}^2 = (c_{14}c_{1,4+1})^2 = c_{15}c_{10} = 1 \rangle$. Then $\theta : \Gamma' \rightarrow G$ is defined by

$$\theta(c_{10}) = x, \quad \theta(c_{11}) = 1, \quad \theta(c_{12}) = y, \\ \theta(c_{13}) = 1, \quad \theta(c_{14}) = z, \quad \theta(c_{15}) = x,$$

verifying $x^2 = y^2 = z^2 = (xy)^4 = (yz)^2 = (zx)^2 = 1$. Hence G is D_4 and calling $z = (xy)^2$ when $D_4 = \langle x, y \mid x^2 = y^2 = (xy)^4 = 1 \rangle$ we achieve the epimorphism.

Since it is easy to prove that C_8 is not G , we take now D_3 . Consider

Γ' an NEC group with signature $(0, +, [2], \{(2, 2, 3)\})$ with presentation $\langle x_{13}, e_1, c_{10}, c_{11}, c_{12}, c_{13} \mid x_1^2 = x_1 e_1 = c_{10}^2 = (c_{10} c_{11})^2 = (c_{11} c_{12})^2 = (c_{12} c_{13})^3 = c_{13} c_{10} = e_1^{-1} c_{10} e_1 c_{13} = 1 \rangle$, and take the epimorphism θ given by

$$\theta(x_1) = \theta(e_1) = yxy, \quad \theta(c_{10}) = x, \quad \theta(c_{11}) = 1, \quad \theta(c_{12}) = x, \quad \theta(c_{13}) = y$$

when $D_3 = \langle x, y \mid x^2 = y^2 = (xy)^3 = 1 \rangle$.

Finally, G may not be C_4 by arguments similar to the above ones.

The three obtained groups are seen to be the full automorphism group of the surface using Teichmüller space techniques as in [3, § 4].

Repeating arguments as in the case above, one gets the following Table II which gives us the complete list of automorphism groups of real algebraic curves of genus 3, according to the topological features :

Table II

$C \setminus C(\mathbf{R})$	k	C_2	$C_2 \times C_2$	D_3	$C_2 \times C_2 \times C_2$	D_4	D_6	$C_2 \times D_4$	S_4
Non-connected	4	*	*		*			*	*
Non-connected	2	*	*	*	*	*	*	*	
Connected	1	*	*			*			
Connected	2	*	*		*	*			
Connected	3	*	*	*		*	*		*

The corresponding results for curves of genus 2 were obtained in [4].

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