# 110. On a Closed Range Property of a Linear Differential Operator

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The purpose of this note is to prove the closed range property of a linear differential operator $P$ acting on the space $\mathcal{A}(K)$ of real analytic functions on a compact subset $K$ of $\boldsymbol{R}^n$ under the condition which we call the uniform $P$-convexity of $K$. Kiro [6] has recently claimed a similar result, but his reasoning contains serious gaps. In connection with this fact, the first named author (T. K.) wants to replace the condition (1.2) in his announcement paper [4] by the condition (1) below. See Kawai [5] for details.

To state our result, let us first prepare some notations. Let $P(x, D_x)$ be a linear differential operator with (not necessarily real-valued) real analytic coefficients defined on an open neighborhood $U$ of $K$. Let $p_m(x, \xi)$ denote the principal symbol of $P(x, D_x)$ and suppose that it has a form $q(x, \xi)^l$ for a positive integer $l$, where $q(x, \xi)$ is a real analytic function in $(x, \xi)$ that is a homogeneous polynomial of $\xi$ of degree $r(=m/l)$. Then the set $K$ is said to be uniformly $P$-convex if $K=\{x \in U; \psi(x) \leqq 0\}$ holds for a real-valued real analytic function $\psi(x)$ which is defined on $U$ satisfying the following condition (1) with some strictly positive constants $A_0$ and $C$:

(1) Setting $z=x+\sqrt{-1}y$ and $\zeta=\dfrac{1}{2}\operatorname{grad}\psi(x)-\sqrt{-1}Ay$, we find

$$\sum_{1\leqq j,k\leqq n}\frac{1}{2}\frac{\partial^2\psi(x)}{\partial x_j \partial x_k}q^{(j)}(z,\zeta)\overline{q^{(k)}(z,\zeta)}+\operatorname{Re}\left(\sum_{j=1}^{n} q_{(j)}(z,\zeta)\overline{q^{(j)}(z,\zeta)}\right)$$
$$-\sum_{j=1}^{n}|q_{(j)}(z,\zeta)|^2/A \geqq C(1+A|y|)^{2(r-1)}$$

for $A>A_0$, on the condition that $q(z,\zeta)=0$ and $A\psi(x)+A^2|y|^2=1$.

Here, and in what follows, $q^{(j)}(z,\zeta)$ (resp., $q_{(j)}(z,\zeta)$) denotes $\partial q/\partial \zeta_j$ (resp., $\partial q/\partial z_j$).

**Remark.** It seems to be interesting that the uniform $P$-convexity is quite akin to the strong $P$-convexity which Hörmander [1] used to obtain a priori estimates of solutions.

Now, our result is the following

**Theorem.** *Let $K$ be a compact subset of $\boldsymbol{R}^n$ and let $P(x, D_x)$ be a linear differential operator defined on an open neighborhood $U$ of $K$. Suppose that $K$ is uniformly $P$-convex. Then $P\mathcal{A}(K)$ is a closed subspace of $\mathcal{A}(K)$.*

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*Proof*. The strategy of our proof is as follows: Let $A$ be a sufficiently large positive number, and let $\varphi_A(z)$ $(z=x+\sqrt{-1}y \in C^n)$ denote $\psi(x)+A|y|^2$. Set $\Omega_A=\{z \in U \times \sqrt{-1}R^n_y ; \varphi_A(z)<1/A\}$. Then we may regard $P$ as a differential operator $P(z, D_z)$ with holomorphic coefficients defined on $\Omega_A$. Let us now denote by $X$ a complexification of $R^n_x \times R^n_y$ and define a $\mathcal{D}_X$-module $\mathcal{M}$ by $\mathcal{D}_X/(\mathcal{D}_X P(z, D_z)+\sum_{j=1}^n \mathcal{D}_X \bar{\partial}_j)$, where $\bar{\partial}_j$ denotes $(\partial/\partial x_j+\sqrt{-1}\partial/\partial y_j)/2$. Note that $P(z, D_z)$ and $\bar{\partial}_j$ commute. Since $\{\Omega_A\}_{A>0}$ is a fundamental system of neighborhoods of $K$, and since $\Omega_A$ is Stein for $A$ sufficiently large, we find

$$\mathcal{A}(K)/P\mathcal{A}(K)=\varinjlim_{A\to\infty} \mathrm{Ext}^1_{\mathcal{D}_X}(\Omega_A ; \mathcal{M}, \mathcal{B}_{R^{2n}_{(x,y)}}). \tag{2}$$

Therefore, if we can prove that the right-hand side of (2) is countable-dimensional, then a result in functional analysis (cf. Komatsu [7], for example) tells us that $P\mathcal{A}(K)$ is a closed subspace of $\mathcal{A}(K)$. This is a sketch of our strategy.

To bring this strategy into practice, we use the results in Kawai [2], [3] on the finite-dimensionality of cohomology groups; we calculate the generalized Levi form of the "positive" tangential system $\mathcal{N}_{A,+}$ on the boundary of $\Omega_A$ induced from $\mathcal{M}$. If we can verify that the generalized Levi form is positive-definite at each characteristic point of $\mathcal{N}_{A,+}$, then $\mathrm{Ext}^1(\Omega_A ; \mathcal{M}, \mathcal{B})$ is finite-dimensional, and hence the right-hand side of (2) is at most countable-dimensional. Since the cotangential component of a characteristic point of $\mathcal{N}_{A,+}$ is determined by its base point $z$ in our case, we denote by $L_z$ the generalized Levi form calculated at the characteristic point in question. The definition of the generalized Levi form is given in [9], Chap. III, Definition 2.3.1, and an explicit form suitable for the present situation is given in [8]. Here, in order to facilitate our calculations, we introduce another Hermitian form $Q_{z_0}(\tau)$ $(\tau \in C^{n+1})$ whose positive-definiteness entails that of $L_{z_0}$. The form $Q_{z_0}(\tau)$ is, by definition, $\sum_{1\leq j,k\leq n+1} a_{jk}(z_0)\tau_j\bar{\tau}_k$, where $a_{jk}(z_0)$ is given as follows:

$$a_{jk}(z_0)=\frac{\partial^2\varphi_A}{\partial z_j \partial \bar{z}_k}(z_0) \qquad (1\leq j, k\leq n), \tag{3}$$

$$\begin{aligned} a_{j,n+1}(z_0)=\overline{a_{n+1,j}(z_0)}=&-\overline{q_{(j)}(z_0, \mathrm{grad}_z\,\varphi_A(z_0))} \\ &-\sum_{1\leq k\leq n}\overline{q^{(k)}(z_0, \mathrm{grad}_z\,\varphi_A(z_0))}\frac{\partial^2\varphi_A}{\partial \bar{z}_j \partial \bar{z}_k}(z_0) \qquad (1\leq j\leq n), \end{aligned} \tag{4}$$

$$\begin{aligned} &a_{n+1,n+1}(z_0) \\ &=\sum_{1\leq j,k\leq n}\frac{\partial^2\varphi_A}{\partial z_j \partial \bar{z}_k}(z_0) q^{(j)}(z_0, \mathrm{grad}_z\,\varphi_A(z_0))\overline{q^{(k)}(z_0, \mathrm{grad}_z\,\varphi_A(z_0))}, \end{aligned} \tag{5}$$

where $z_0$ satisfies $\varphi_A(z_0)=1/A$ and $q(z_0, \mathrm{grad}_z\,\varphi_A(z_0))=0$.

To use this formula, let us first note the following two facts: First, each principal minor of the matrix $\alpha(z_0) \underset{\mathrm{def}}{=} (a_{jk}(z_0))_{1\leq j,k\leq n+1}$ that does not intersect with its $(n+1)$-th row is positive for $A$ sufficiently large. Hence it suffices for us to verify the positivity of $\det(\alpha(z_0))$. The second fact we note is that the uniform $P$-convexity is invariant under a real orthogonal transformation. That is, if we define $\tilde{z}=\tilde{x}+\sqrt{-1}\tilde{y}$ by $\tilde{z}=M^{-1}(x-\operatorname{Re} z_0+\sqrt{-1}y)$ for a real orthogonal matrix $M$, then the uniform $P$-convexity holds for the new variable $\tilde{z}$.

Now, to calculate $\det(\alpha(z_0))$, let us choose an orthogonal matrix $M$ that brings $((\partial^2\psi/\partial x_j\partial x_k)(\operatorname{Re} z_0))_{1\leq j,k\leq n}$ to a diagonal matrix in the coordinate system $(\tilde{x}, \tilde{y})$ defined above. Then we find

$$(6)\quad \det(\alpha(\tilde{z}_0))=\prod_{j=1}^{n}(c_j+\tilde{A})\Big(\sum_{1\leq j\leq n}(c_j+\tilde{A})|q^{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))|^2\Big)$$
$$-\sum_{1\leq j\leq n}\frac{\prod_{k=1}^{n}(c_k+\tilde{A})}{c_j+\tilde{A}}|q_{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))+(c_j-\tilde{A})q^{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))|^2,$$

where

$$c_j=\frac{1}{4}\frac{\partial^2\psi}{\partial\tilde{x}_j^2}(\tilde{x}_0),\ \tilde{A}=\frac{1}{2}A \quad\text{and}\quad \tilde{z}_0=\sqrt{-1}M^{-1}(\operatorname{Im} z_0).$$

Hence, by setting $(1+A|\tilde{y}_0|)=\rho$, we obtain

$$(7)\quad \det(\alpha(\tilde{z}_0))=\Big\{\sum_{1\leq j\leq n}4c_j|q^{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))|^2$$
$$+2\operatorname{Re}\Big(\sum_{1\leq j\leq n}\overline{q_{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))}q^{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))\Big)$$
$$-\sum_{1\leq j\leq n}|q_{(j)}(\tilde{z}_0, \operatorname{grad}_{\tilde{z}}\varphi_A(\tilde{z}_0))|^2/\tilde{A}\Big\}\tilde{A}^n+R(\tilde{z}_0),$$

where

$$(8)\quad |R(\tilde{z}_0)|\leq C'\tilde{A}^n\rho^{2(r-1)}(1+\rho\tilde{A}^{-1})\rho\tilde{A}^{-1}$$

holds for a constant $C'$. Since $\varphi_A(\tilde{z}_0)=1/A$ holds by the definition, $\rho\tilde{A}^{-1}$ tends to zero as $A$ tends to infinity. Therefore the condition (1) guarantees that $\det(\alpha(\tilde{z}_0))$ is positive for sufficiently large $A$. This completes the proof. Q.E.D.

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