107. *A Characterization of Chebyshev Spaces*

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§1. **Introduction.** Let M be a finite dimensional linear subspace of $C[a,b]$, the space of real valued continuous functions defined on a finite closed interval $[a,b]$. Then, for a function $f\in C[a,b]$, we are concerned with the approximation problem:

$$\text{find } \tilde{f}\in M \text{ to minimize } \|f-\tilde{f}\|,$$

where $\|\cdot\|$ denotes the uniform norm. The function $\tilde{f}\in M$ is said to be a best approximation to f from M if $\tilde{f}$ is a solution to the above problem. For an n-dimensional subspace M, we put the following two subsets of $C[a,b]$: $U_M=\{f\,|\,f$ possesses a unique best approximation$\}$ and $A_M=\{g\,|\,$the error function $e=g-\tilde{g}$ has an alternating set of $(n+1)$ points in $[a,b]$ for any best approximation $\tilde{g}$ to g; i.e., there exist $(n+1)$ distinct points $a\leqq x_1<\cdots<x_{n+1}\leqq b$ such that $|e(x_i)|=\|e\|$, $i=1,2,\cdots,n+1$ and $e(x_i)\cdot e(x_{i+1})\leqq 0$, $i=1,\cdots,n\}$.

As is well known, if M is a Chebyshev space (respectively weak Chebyshev space), that is, every nonzero function in M has no more than $n-1$ zeros (respectively changes of sign) on $[a,b]$, then they are of great use in this problem. Hence various properties and characterizations of these spaces have been obtained. Young [5] showed that if M is a Chebyshev space then U_M is equal to $C[a,b]$. Further, by the result of Haar [1], a necessary and sufficient condition that M is a Chebyshev space is that U_M coincides with $C[a,b]$.

As a characterization of a weak Chebyshev space, Jones and Karlovitz [2] proved that M is a weak Chebyshev space if and only if U_M is included in A_M. In this paper, as the above result, we shall give a characterization of a Chebyshev space M by using an inclusion relation between U_M and A_M.

§2. **Definitions and lemmas.** In this section, we prepare several lemmas necessary for the proof of the main theorem. First we begin with some definitions.

Definition 1. For a function $f\in C[a,b]$, two zeros x_1,x_2 of f are said to be *separated* if there is an x_0, $x_1<x_0<x_2$, such that $f(x_0)\neq 0$.

For an n-dimensional subspace M of $C[a,b]$, we define the followings.

Definition 2. (i) We call a point $x_0\in[a,b]$ *vanishing* with respect to M if $g(x_0)=0$ for any $g\in M$. In case that no confusion arises, the term "with respect to M" will be omitted.

(ii) M is called *vanishing* if there exists at least one vanishing point in $[a,b]$. Otherwise, it is called *nonvanishing*.

Definition 3. M is said to have (*)-*property* if a function $g\in M-\{0\}$ vanishes identically on a nondegenerate subinterval of $[a, b]$.

Let G be an n-dimensional weak Chebyshev space of $C[a, b]$. Then we can show the following three lemmas which are of independent interest.

Lemma A (Stockenberg [4]). (i) *If there is a $g\in G$ with n separated, nonvanishing zeros $a\leqq x_1<\cdots<x_n\leqq b$, then $g(x)=0$ for all $x\in[a, x_1]\cup[x_n, b]$.*

(ii) *No $g\in G$ has more than n separated, nonvanishing zeros.*

Lemma B. *Suppose that G does not have (*)-property. Suppose also that G contains a strictly positive function and contains two functions $r, s\in G$ such that*

$$\det\begin{pmatrix} r(a) & r(b)\\ s(a) & s(b)\end{pmatrix}\neq 0.$$

Then G is a Chebyshev space.

We denote by $G|_{[c,d]}$ the space obtained by restricting G to a subinterval $[c, d]$ of $[a, b]$.

Lemma C (Sommer [3]). *If $a\leqq c<d\leqq b$, then the space $G|_{[c,d]}$ is a weak Chebyshev space of $C[c, d]$ with dimension less or equal to n.*

Remark 1. From Theorem 1 and Theorem 4 in Stockenberg [4], Lemma B follows immediately.

§3. Main theorem. Let M be an n-dimensional linear subspace of $C[a, b]$. We give the result due to Jones and Karlovitz [2] again.

Theorem A. *M is a weak Chebyshev space if and only if $A_M\supset U_M$.*

Now we can establish the following

Theorem. *M is a Chebyshev space if and only if $A_M=U_M\cup L$, where L denotes the set of all real-valued linear functions on $[a, b]$.*

Proof. In one direction, this is trivial. Hence it is sufficient to verify that M is a Chebyshev space under the assumption that $A_M=U_M\cup L$.

First we show that M is a weak Chebyshev space containing a strictly positive function. By Theorem A, it is clear that M is weak Chebyshev. Provided that M does not contain a strictly positive function, then one of the best approximations to the constant function $1\in L$ from M is 0. But this contradicts the assumption. Hence, in case that $n=1$, M is Chebyshev. In the rest of the proof, we assume $n\geqq 2$.

Next we show that M does not have (*)-property. Suppose that there exists a function $f\in M-\{0\}$ vanishing identically on a nondegenerate subinterval $[c, d]$ of $[a, b]$, where $a\leqq c<d\leqq b$. Then it follows from the fact that M contains a strictly positive function and Lemma C that $M_1=M|_{[c,d]}$ obtained by restricting M to a subinterval $[c, d]$ is a nonvanishing weak Chebyshev space such that $\dim M|_{[c,d]}<n$. In case that M_1 has (*)-property, we can also consider a nonvanishing weak Chebyshev space $M_1|_{[\alpha,\beta]}$ obtained by the same way with respect to M_1, where $c\leqq\alpha<\beta\leqq d$ and $\dim M_1|_{[\alpha,\beta]}<\dim M_1$. Since M contains a strictly positive function, by continuing the above procedure at most $n-1$ times, we consequently obtain a nonvanishing weak Chebyshev space $M|_{[\gamma,\delta]}$ without (*)-property, where $c \leqq \gamma < \delta \leqq d$ and $m = \dim M|_{[\gamma,\delta]} < n$. Now we consider a function f_0, which is satisfied with the following conditions:

(i) $f_0(x) = 0$ for $x \in [a, \gamma] \cup [\delta, b]$.

(ii) There are $2(n+m+2)$ points $\gamma < z_1 < \cdots < z_{2(n+m+2)} < \delta$ of (γ, δ) such that $|f_0(z_i)| = \|f_0\| > 0$, $i = 1, 2, \cdots, 2(n+m+2)$ and $f_0(z_i) \cdot f_0(z_{i+1}) < 0$ for $i = 1, 2, \cdots, 2n+2m+3$. Since M is assumed to have (*)-property, there is such a function $h^* \in M - \{0\}$ that $\|h^*\| < \|f_0\|$ and $h^*(x) = 0$ for $x \in [c, d]$. Then each function $\lambda \cdot h^*$, $0 \leqq \lambda \leqq 1$, is a best approximation to f_0 from M because M is weak Chebyshev and the error function $f_0 - \lambda \cdot h^*$ has an alternating set of $(n+1)$-points in $[a, b]$. On the other hand, since $M|_{[\gamma,\delta]}$ is a nonvanishing weak Chebyshev space without (*)-property, by Lemma A, we can see that each function $f \in M|_{[\gamma,\delta]} - \{0\}$ has at most m zeros in $[\gamma, \delta]$. Providing that there exists a best approximation to f_0 from M which has an alternating set of at most n-points in $[\gamma, \delta]$, then it has at least $(m+1)$ zeros in $[\gamma, \delta]$. This leads to a contradiction. Eventually, by these facts, we conclude that f_0 is contained in A_M but not in $U_M \cup L$, which is the contrary to the assumption.

Finally we show that M contains two functions $r, s \in M$ such that

$$\det \begin{pmatrix} r(a) & r(b) \\ s(a) & s(b) \end{pmatrix} \neq 0.$$

Let r be a strictly positive function whose existence is guaranteed in the first half of this proof. As a function s, we choose a best approximation to the linear function $l(x) = -x + 1/2$. Noting that $l - s$ has an alternating set of at least 3-points in $[a, b]$, it holds that $s(a) \cdot s(b) < 0$, because s can not be a best approximation to l in the other cases. Thus we have

$$\det \begin{pmatrix} r(a) & r(b) \\ s(a) & s(b) \end{pmatrix} \neq 0.$$

Consequently, from Lemma B, it follows that M is a Chebyshev space.

Corollary. *Let G be an n-dimensional nonvanishing weak Chebyshev space of $C[a, b]$. Then G has (*)-property if and only if $A_G \supsetneqq U_G$.*

Proof. First suppose that G has (*)-property. By using the proof of Theorem, we easily observe that $A_G \supsetneqq U_G$.

Next suppose that any nonzero function in G has at most n zeros, because, by Lemma A, this is equivalent to the fact that G does not have (*)-property. For any function $g \in A_G$, let r, s be best approximations to g from G. Since the function $(r+s)/2$ is also a best approximation to g, $g - (r+s)/2$ has an alternating set of $(n+1)$ points $\{z_i\}_{i=1}^{n+1}$ in $[a, b]$. Hence, for these points $\{z_i\}_{i=1}^{n+1}$, we have

$$\begin{aligned} \|g - (r+s)/2\| &= |g(z_i) - (r(z_i) + s(z_i))/2| \\ &\leqq (1/2) \cdot \{|g(z_i) - r(z_i)| + |g(z_i) - s(z_i)|\} \\ &\qquad i = 1, 2, \cdots, n+1, \end{aligned}$$

which means that

$$g(z_i) - r(z_i) = g(z_i) - s(z_i) \qquad i = 1, 2, \cdots, n+1.$$

Thus $r-s$ has at least $(n+1)$ zeros, which leads to the fact that r is identical with s on $[a, b]$. Hence we have $A_G=U_G$. It completes the proof.

Remark 2. (1) An important example fitting the condition in Corollary is given by a polynomial spline function space with fixed knots, and some examples which are not Chebyshev spaces without (*)-property are shown in Stockenberg [4].

(2) The assertion in Corollary does not always hold under the assumption having finite vanishing points instead of no vanishing points with respect to the space G. For instance, on $C[0, \pi]$, let us consider the space $G=\{\lambda\cdot\sin x \mid \lambda\in R\}$. Clearly G is a weak Chebyshev space which does not have (*)-property but 2 vanishing points in $[0, \pi]$. Then the best approximation to the linear function $f(x)=-2x+\pi$ is not unique and any best approximation to it has an alternating set of 2 points, 0 and π. Thus we obtain $A_G\supsetneqq U_G$. Moreover, generalizing this example, we can easily show that $A_G\supsetneqq U_G$ for every n-dimensional weak Chebyshev space G with more than $(n+1)$ vanishing points, which consists of continuously differentiable functions.

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