102. On Automorphisms of Algebraic K3 Surfaces which Act Trivially on Picard Groups

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(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1986)

1. Introduction. In this note we study automorphisms of algebraic $K3$ surfaces over $\boldsymbol{C}$ which act trivially on Picard groups. Recall that a $K3$ *surface* X is a nonsingular compact complex surface with trivial canonical bundle and $\dim H^1(X, \mathcal{O}_X)=0$. The second cohomology group $H^2(X, \boldsymbol{Z})$ admits a canonical structure of a lattice of rank 22 induced from the cup product. We denote by S_X the Picard group of X. Then S_X has a structure of a sublattice of $H^2(X, \boldsymbol{Z})$. Let T_X be the orthogonal complement of S_X in $H^2(X, \boldsymbol{Z})$ which is called a *transcendental lattice* of X. Put $H_X=\mathrm{Ker}(\mathrm{Aut}(X)\to\mathrm{Aut}(S_X))$. Then H_X is a cyclic group $\boldsymbol{Z}/m$ of order m, and $\phi(m)$ is a divisor of the rank of T_X where ϕ is the Euler function ([3], Corollary 3.3).

Theorem. *Let X be an algebraic $K3$ surface and m_X the order of H_X. Assume that the lattice T_X is unimodular* (*i.e.* $\det(T_X)=\pm 1$). *Then*

(1) *m_X is a divisor of* 66, 44 *or* 12.

(2) *Suppose that* $\phi(m)=\mathrm{rank}(T_X)$. *Then m_X is equal to either* 66 *or* 42. *Moreover for $m=66$ or 42, there exists a unique* (*up to isomorphisms*) *algebraic $K3$ surface with $m_X=m$.*

In case T_X is non unimodular, Vorontsov [8] proved a similar result as the above theorem. However his statement for unimodular case is not complete and contains a mistake, i.e. he claims that there exists an algebraic $K3$ surface with $m_X=12$ and $\mathrm{rank}(T_X)=\phi(12)$ (his proof has not yet published). His method is based on the theory of a cyclotomic field $\boldsymbol{Q}(m)$. Here we use only the theory of elliptic surfaces due to Kodaira [1].

2. Example. In this section we construct two examples of algebraic $K3$ surfaces with $m_X=66$, 42.

(2.1) **Example 1.** Let (x, y, z) be a system of a homogeneous coordinate of $\boldsymbol{P}^2$. We take two copies $W_0=\boldsymbol{P}^2\times C_0$ and $W_1=\boldsymbol{P}^2\times C_1$ of the cartesian product $\boldsymbol{P}^2\times\boldsymbol{C}$ and form their union $W=W_0\cup W_1$ by identifying $(x, y, z, u)\in W_0$ with $(x_1, y_1, z_1, u_1)\in W_1$ if and only if $u\cdot u_1=1$, $x=x_1$, $y=u_1^6\cdot y$ and $z=u_1^2\cdot z_1$. We define a subvariety X of W by the following equations:

$$(2.2)\qquad \begin{aligned} &z^3-y\left\{y^2\prod_{i=1}^{12}(u-\xi_i)-x^2\right\}=0,\\ &z_1^3-y_1\left\{y_1^2\prod_{i=1}^{12}(1-u_1\cdot\xi_i)-x_1^2\right\}=0 \end{aligned}$$

where ξ_i $(i=1, 2, \cdots, 12)$ are distinct comlex numbers. Let π be a projection from X to the u-sphere $\boldsymbol{P}^1$. It is easy to see that X is non singular and $\pi^{-1}(u)$ is a non singular elliptic curve with the functional invariant zero for every u except ξ_i $(i=1, \cdots, 12)$. Moreover we can see that $\pi^{-1}(\xi_i)$ is a singular fibre of type II, namely a rational curve with one cusp, and X is a $K3$ surface. The curve $L=\{y=z=0\}=\{y_1=z_1=0\}$ gives a holomorphic section of the elliptic pencil π. And also the form $\omega=w$ $'du\wedge dv$ gives a nowhere vanishing holomorphic 2-form on X where $(u, w=z/x, v=y/x)$ is an affine coordinate. The above construction of X is due to Shiga [6], Remark 1–3 (also see [2]). We define an automorphism g_1 of X as follows: $g_1(x, y, z, u)=(-x, y, e_3\cdot z, u)$, $g_1(x_1, y_1, z_1, u_1)=(-x_1, y_1, e_3\cdot z_1, u_1)$ where e_3 is a primitive 3-th root of unity. Obviously g_1 is of order 6.

In the following we assume that $\xi_{12}=0$ and $\xi_i=e_{11}^i$ $(i=1, \cdots, 11)$ where e_{11} is a primitive 11-th root of unity. Then $g_2(x, y, z, u)=(x, e_{11}^8\cdot y, e_{11}^{10}\cdot z, e_{11}^6\cdot u)$, $g_2(x_1, y_1, z_1, u_1)=(x_1, y_1, z_1, e_{11}^5\cdot u_1)$ defines an automorphism of X of order 11. Put $g=g_1\circ g_2=g_2\circ g_1$. Then g is of order 66 and $g^*\omega=-e_3\cdot e_{11}^5\cdot\omega$. Since $\phi(66)|\mathrm{rank}(T_X)$, we have $\mathrm{rank}(T_X)=20$. Hence $\mathrm{rank}(S_X)=2$ (recall that $\mathrm{rank}\,H^2(X, \boldsymbol{Z})=22$). Note that S_X contains both classes of a fibre of π and the section L which form a unimodular lattice $U=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ of rank 2. Hence S_X is isomorphic to U. Since T_X is the orthogonal complement of S_X in the unimodular lattice $H^2(X, \boldsymbol{Z})$, T_X is also unimodular (cf. [3], § 1).

(2.3) **Remark.** In the equations (2.2), put $\xi_i=e_{12}^i$ $(i=1, \cdots, 12)$ where e_{12} is a primitive 12-th root of unity. Then we obtain an algebraic $K3$ surface with $m_X=12$ and $S_X=U$.

(2.4) **Example 2.** With the same notation as in Example 1, we define a subvariety Y' of W by the following equations:

$$z_3-y\{y^2(u-\xi_0)^5\textstyle\prod_{i=1}^7(u-\xi_i)-x^2\}=0,$$
$$z_1^3-y_1\{y_1^2(1-u\xi_0)^5\textstyle\prod_{i=1}^7(1-u\xi_i)-x_1^2\}=0.$$

It is easy to see that Y' has a singularity of type E_8 at $(0, 1, 0, \xi_0)$. Let Y be a minimal resolution of Y'. Then Y is a $K3$ surface. Let $\pi: Y\to\boldsymbol{P}^1$ be a map induced from a projection from Y' to the u-sphere $\boldsymbol{P}^1$. We can see that $\pi^{-1}(u)$ is a non singular elliptic curve with the functional invariant zero for every u except ξ_i $(i=0, 1, \cdots, 7)$. Moreover $\pi^{-1}(\xi_0)$ is a singular fibre of type II* and $\pi^{-1}(\xi_i)$ is a singular fibre of type II $(i=1, \cdots, 7)$. Now we put $\xi_0=0$ and $\xi_i=e_7^i$ $(i=1, \cdots, 7)$ where e_7 is a primitive 7-th root of unity. Then in the similar way as in Example 1, we can construct an automorphism g of order 42. It is easy to see that T_X is isomorphic to a unimodular lattice $U\oplus U\oplus E_8$ where E_8 is a negative definite lattice of rank 8 associated with the Dynkin diagram of type E_8. From the construction, g^* acts on S_X as identity.

3. Proof of Theorem. First we recall that T_X is isomorphic to $U\oplus U$, $U\oplus U\oplus E_8$ or $U\oplus U\oplus E_8\oplus E_8$ because T_X is an even unimodular lattice (cf. [5]). Hence S_X is isomorphic to $U\oplus E_8\oplus E_8$, $U\oplus E_8$ or U, respectively. The following Lemma follows from [4], § 3, Corollary 3 and the classification of singular fibres of elliptic pencils [1].

(3.1) **Lemma.** *X has an elliptic pencil π with a section. Its only reducible singular fibre (if exists) is of type* II*.

(3.2) *Proof of the assertion* (2). In case $T_X = U \oplus U$, then $m_X = 12$, 10 or 8. Since $S_X = U \oplus E_8 \oplus E_8$, the elliptic pencil π has two reducible singular fibres of type II*, and other singular fibres are either of type II or of type I_1. We denote by r, resp. s, the number of singular fibres of type II, resp. type I_1. Then by the formula [1], (12.6), we have $2r+s=4$. Note that any g ($g \in H_X$) preserves the structure of the pencil π and a section of π, and hence the order of the restriction of g on fibres is a divisor of 6 or 4. If g is of order 12, then we can see that $(r, s) = (2, 0)$ and the order of the restriction of g on fibres is 6. However this is impossible since g^6 acts on X as identity. Similarly we conclude $m_X \neq 12$, 10 and 8.

In the same way, we have $m_X = 66$ if $T_X = U \oplus U \oplus E_8 \oplus E_8$ and $m_X = 42$ or 26 if $T_X = U \oplus U \oplus E_8$. Moreover if $m_X = 66$, then the order of the restriction of H_X on fibres is divisible by 3 and hence the functional invariant of π is a constant $(=0)$. Hence all singular fibres of π are of type II. Similarly if $m_X = 42$, then π has one singular fibre of type II* and 7 singular fibres of type II. We now claim that $m_X = 26$ does not occur. If g is an automorphism of order 26 ($g \in H_X$), then π has 14 singular fibres of type I_1. g fixes one singular fibre F of type I_1 and acts on the set of other 13 singular fibres of type I_1 as a permutation of order 13. Since g preserves a node p of F and a section of π, F is a fixed curve of g^2. Hence g^2 acts on the tangent space of X at p as identity. This is a contradiction because $(g^2)^* \omega_X = e_{13} \cdot \omega_X$ where ω_X is a nowhere vanishing holomorphic 2-form of X and e_{13} is a primitive 13-th root of unity.

(3.3) **Uniqueness of $K3$ surfaces with $m_X = 66$, 42.** Let X be an algebraic $K3$ surface with $m_X = 66$. We have already seen that such $K3$ surface exists (§ 2). By the above observation (3.2), X must have an elliptic pencil $\pi: X \to \boldsymbol{P}^1$ with a section L which has 12 singular fibres of type II. Denote by $\{\xi_i\}$ the set of points of $\boldsymbol{P}^1$ such that $\pi^{-1}(\xi_i)$ is singular ($i = 0, 1, \cdots, 11$). We may assume that g fixes ξ_0 and acts on $\{\xi_1, \cdots, \xi_{11}\}$ as a permutation. Also g induces an automorphism of order 6 on fibres of π. Now we take a homology basis of $H_2(X, \boldsymbol{Z})$ as follows (see [6], § 2): Let F be a smooth fibre of π and $\{\gamma_1, \gamma_2\}$ a basis of $H_1(F, \boldsymbol{Z})$. And let α_i ($i = 1, 2, \cdots, 10$) be an oriented arc in $\boldsymbol{P}^1$ which starts from ξ_0 and goes to ξ_i so that α_i does not intersect any other α_j. We set

$$
\begin{aligned}
&C_{2i-1} = \alpha_i \times \gamma_1, \\
&C_{2i} = \alpha_i \times \gamma_2 \qquad \text{for } i = 1, \cdots, 10, \\
&C_{21} = F, \\
&C_{22} = L.
\end{aligned}
$$

Then $\{C_1, \cdots, C_{22}\}$ gives a basis of $H_2(X, \boldsymbol{Z})$ ([6], Proposition 2-1). The action of g_* on $H_2(X, \boldsymbol{Z})$ is unique up to $\mathrm{Aut}(H_2(X, \boldsymbol{Z}))$. Note that a nowhere vanishing holomorphic 2-form on X is an eigenvector of g^* acting on $H^2(X, \boldsymbol{C})$. Hence the uniqueness of algebraic $K3$ surface with $m_X = 66$

easily follows from the Torelli theorem for algebraic $K3$ surfaces ([4]). The same observation shows the uniqueness of algebraic $K3$ surface with $m_X=42$. We omit the proof.

(3.4) *Proof of the assertion* (1). The same argument as in (3.2) shows that m_X is a divisor of 66, 42 or 12 except in the following two cases: $S_X=U$ and $5|m_X$ or $m_X=8$. In any case there exists an automorphism g of X which acts on $\boldsymbol{P}^1$ as a permutation of order 5 or 2. However it follows from the Lefschetz fixed point formula [7], Lemma 1.6 that these cases do not occur. In fact the Lefschetz number of g is equal to $4-20/\phi(|g|)$ which is negative integer. On the other hand, the fixed curves of g are contained in fibres of π, and hence their Euler numbers are non negative, which is a contradiction.

Added in Proof. I. Dolgachev and T. Shioda have informed the author that they gave another simple construction of algebraic $K3$ surfaces with $m_X=66$, 42 and 12.

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