70. *Asymptotic Behavior of Solutions for the Equations of a Viscous Heat-conductive Gas*

By Shuichi KAWASHIMA,*) Akitaka MATSUMURA,**) and Kenji NISHIHARA***)

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1. Introduction. We study the asymptotic behavior of solutions to the initial value problem for the equations of a viscous heat-conductive gas in Lagrangian coordinates:

$$(1)\qquad \begin{aligned} &v_t - u_x = 0, \qquad u_t + p_x = (\mu u_x / v)_x, \\ &(e + u^2/2)_t + (pu)_x = (\kappa\theta_x / v + \mu u u_x / v)_x, \end{aligned}$$

where the unknowns $v>0$, u and $\theta>0$ represent the specific volume, the velocity and the absolute temperature of the gas. The coefficients of viscosity and heat-conductivity, μ and κ, are assumed to be positive constants. The pressure p, the internal energy e and the entropy s are smooth functions of (v, θ). Also, p and e are regarded as smooth functions of (v, s). We write $p = p(v, \theta) = \hat{p}(v, s)$, $e = e(v, \theta) = \hat{e}(v, s)$, $s = s(v, \theta)$ and assume that $\partial p(v, \theta)/\partial v < 0$, $\partial e(v, \theta)/\partial \theta > 0$, $\partial^2 \hat{p}(v, s)/\partial v^2 > 0$ and $\hat{p}(v, s)$ is a convex function of (v, s). These conditions together with the thermodynamic relation $de = \theta ds - p dv$ ensure that the corresponding inviscid system

$$(2)\qquad v_t - u_x = 0, \quad u_t + p_x = 0, \quad (e + u^2/2)_t + (pu)_x = 0$$

is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate ([2]).

We denote the initial function for (1) by $U_0(x) = (v_0, u_0, \theta_0)(x)$ and put $U_\pm = U_0(\pm\infty)$. When $U_- = U_+$, it was shown in [6] that the solution of (1) converges to the constant state $U_- = U_+$ as $t \to \infty$. The case $U_- \neq U_+$ was studied recently in [4], [1], [3] under the hypothesis that U_- is connected to U_+ by only shock waves. It was proved that the solution of (1) approaches the superposition of smooth traveling waves with shock profile. In this paper, we consider the case where U_- is connected to U_+ by only rarefaction waves, and show that the solution of (1) converges to the weak solution of the Riemann problem for the inviscid equations (2). A similar result has been obtained in [5] for the barotropic model gas.

2. Theorems. In what follows, we assume that $\delta = |U_+ - U_-|$ is small and U_- is connected to U_+ by only rarefaction waves. We denote by $\bar{U}(t, x) = (\bar{v}, \bar{u}, \bar{\theta})(t, x)$ the weak solution to the Riemann problem for (2) with the step initial data $\bar{U}_0(x) = (\bar{v}_0, \bar{u}_0, \bar{\theta}_0)(x) = U_\pm$, $x \gtrless 0$ (cf. [2]). Our main result is stated as follows.

*) Department of Mathematics, Nara Women's University.
**) Department of Applied Mathematics and Physics, Kyoto University.
***) Tokyo National Technical College, Hachioji, Tokyo.

Theorem 1 (general gas). *Assume* $U_0-\bar{U}_0\in L^2$ *and* $\partial_x U_0\in L^2$ *for the initial function* $U_0(x)$. *Then there exist positive constants* δ_0 *and* ε_0 *such that if* $\delta=|U_+-U_-|\le\delta_0$ *and* $E_0=\|U_0-\bar{U}_0\|+\|\partial_x U_0\|\le\varepsilon_0$ ($\|\cdot\|$ *denotes the usual* L^2*-norm), then the initial value problem for* (1) *has a unique global solution* $U(t,x)=(v,u,\theta)(t,x)$ *satisfying* $U-\bar{U}_0\in C^0([0,\infty);L^2)$, $\partial_x U\in C^0([0,\infty);L^2)$ *and* $\partial_x^2(u,\theta)\in L^2([0,\infty);L^2)$. *Moreover,* $U(t,x)$ *converges to the weak solution* $\bar{U}(t,x)$ *uniformly in* $x\in\boldsymbol{R}$ *as* $t\to\infty$.

Next, we consider the special case of an ideal polytropic gas, where p and e are given explicitly by $p=R\theta/v=\hat{R}v^{-\gamma}e^{(\gamma-1)s/R}$ and $e=R\theta/(\gamma-1)+$constant. Here $R>0$ is the gas constant, $\gamma>1$ the adiabatic exponent and $\hat{R}$ is a positive constant. Letting $\gamma_0\ge 2$ be an arbitrarily fixed constant, we regard γ as a parameter valued in $(1,\gamma_0]$ and assume that for any fixed positive constants E_1 and m_1, $\|(v_0-\bar{v}_0, u_0-\bar{u}_0, (\theta_0-\bar{\theta}_0)/\sqrt{\gamma-1})\|+\|\partial_x(v_0,u_0,\theta_0/\sqrt{\gamma-1})\|\le E_1$ and $\inf v_0(x)$, $\inf\theta_0(x)\ge m_1$ hold uniformly in $\gamma\in(1,\gamma_0]$. Then we have

Theorem 2 (ideal polytropic gas). *Assume the above conditions for the initial function* $U_0(x)=(v_0,u_0,\theta_0)(x)$. *Then there exist positive constants* δ_1 *and* $\gamma_1\in(1,\gamma_0]$ *depending only on* E_1 *and* m_1 *such that if* $\delta=|U_+-U_-|\le\delta_1$, *the initial value problem for* (1) *has a unique global solution for each* $\gamma\in(1,\gamma_1]$, *which satisfies the same properties as in Theorem* 1.

3. Smooth approximation to the weak solution. To prove the theorems, we employ the technique of [5] and construct a smooth approximating function for the weak solution $\bar{U}(t,x)$. The characteristic roots of (2) are given by $\lambda_1=-(-\hat{p}_v)^{1/2}$, $\lambda_2=0$ and $\lambda_3=(-\hat{p}_v)^{1/2}$, where $\hat{p}_v=\partial\hat{p}(v,s)/\partial v<0$. The first and the third characteristic fields are genuinely nonlinear while the second is linearly degenerate. We denote by $R_j(\underline{U})$ the j-th rarefaction curve through $\underline{U}$, $j=1,3$. Our assumption for $U_\pm$ implies that there exists an intermediate state U_m such that $U_m=R_1(U_-)$ and $U_+=R_3(U_m)$. Therefore the weak solution $\bar{U}(t,x)$ can be decomposed as $\bar{U}(t,x)=\bar{U}^1(t,x)+\bar{U}^3(t,x)-U_m$, where each $\bar{U}^j(t,x)$ is determined by $\bar{U}^j(t,x)\in R_j(U_-^j)$ and $\lambda_j(\bar{U}^j(t,x))=\bar{z}^j(t,x)$. Here $\bar{z}^j(t,x)$ is the weak solution of the inviscid Burgers equation $z_t+zz_x=0$ with the step initial data $\bar{z}_0^j(x)=z_\pm^j\equiv\lambda_j(U_\pm^j)$, $x\gtrless 0$. (Here we write $U_-=U_-^1$, $U_m=U_+^1=U_-^3$, $U_+=U_+^3$.)

As in [5], we approximate the step function $\bar{z}_0^j(x)$ by the smooth function $\tilde{z}_0^j(x)=(1/2)\{(z_+^j+z_-^j)+(z_+^j-z_-^j)\tanh x\}$. Let $\tilde{z}^j(t,x)$ be the corresponding smooth solution of the inviscid Burgers equation. We construct $\tilde{U}^j(t,x)$ by $\tilde{U}^j(t,x)\in R_j(U_-^j)$ and $\lambda_j(\tilde{U}^j(t,x))=\tilde{z}^j(t,x)$, and then put $\tilde{U}(t,x)=\tilde{U}^1(t,x)+\tilde{U}^3(t,x)-U_m$. By the definition, $\tilde{U}(t,x)$ converges to the weak solution $\bar{U}(t,x)$ uniformly in $x\in\boldsymbol{R}$ as $t\to\infty$. We know also that $\tilde{U}(t,x)=(\tilde{v},\tilde{u},\tilde{\theta})(t,x)$ satisfies $\tilde{u}_x\ge 0$ and

(3) $$\tilde{v}_t-\tilde{u}_x=0,\quad \tilde{u}_t+\tilde{p}_x=f_x,\quad (\tilde{e}+\tilde{u}^2/2)_t+(\tilde{p}\tilde{u})_x=\tilde{u}f_x,$$

where $\tilde{p}=p(\tilde{v},\tilde{\theta})$ etc., and $f(t,x)$ is a rapidly decreasing function of $(t,x)\in[0,\infty)\times\boldsymbol{R}$. Moreover, for $p\in[1,\infty]$, we have the estimates $\|\partial_x^l\tilde{U}(t)\|_{L^p}\le C\delta$, $l=1,2$, $\|\partial_x\tilde{U}(t)\|_{L^p}\le C\delta^{1/p}(1+t)^{-(1-1/p)}$ and $\|\partial_x^2\tilde{U}(t)\|_{L^p}\le C(1+t)^{-1}$, where

$\delta=|U_+-U_-|$ and C is a positive constant (cf. [5]).

4. Outline of the proof of theorems. We seek the solution of (1) in the form $U(t,x)=\tilde{U}(t,x)+\Psi(t,x)$ with $\Psi\in X([0,\infty))$. Here $X(I)$ (I is an interval in $[0,\infty)$) denotes the set of all functions $\Psi(t,x)=(\phi,\psi,\zeta)(t,x)$ satisfying $\Psi\in C^0(I;H^1)$, $\partial_x\phi\in L^2(I;L^2)$, $\partial_x(\psi,\zeta)\in L^2(I;H^1)$ and $\inf v(t,x)$, $\inf\theta(t,x)>0$, where inf is taken over $I\times \boldsymbol{R}$ and $v(t,x)=\tilde{v}(t,x)+\phi(t,x)$ etc. Using (3) we rewrite (1) to get the system for Ψ and then consider the resulting system with the initial condition $\Psi(\tau,x)=\Psi_\tau(x)=(\phi_\tau,\psi_\tau,\zeta_\tau)(x)$ for each $\tau\geq 0$. It is proved by the standard iteration method that if $\Psi_\tau\in H^1$ and $\inf v_\tau(x)$, $\inf\theta_\tau(x)>0$ hold uniformly in $\tau\geq 0$, then the problem has a unique solution $\Psi\in X([\tau,\tau+T_0])$ for a positive constant T_0 independent of $\tau\geq 0$. Here $v_\tau(x)=\tilde{v}(\tau,x)+\phi_\tau(x)$ etc. Therefore, to prove our theorems, it suffices to get desired a priori estimates for the solution $U(t,x)$ of (1) satisfying $\Psi=U-\tilde{U}\in X([0,T])$.

Proposition 3 (general gas). *Let $U(t,x)$ be a solution of* (1) *in the sense stated above. Assume that $\|\Psi(t)\|_1\leq \underline{E}$, $t\in[0,T]$, and $v(t,x)$, $\theta(t,x)\geq \underline{m}$, $(t,x)\in[0,T]\times\boldsymbol{R}$, for positive constants $\underline{E}$ and $\underline{m}$, where $\|\cdot\|_1$ denotes the H^1-norm. Then there are positive constants δ_2 and C not depending on T such that if $\delta=|U_+-U_-|\leq\delta_2$, then*

$$\sup_{0\leq t\leq T}\|\Psi(t)\|_1^2+\int_0^T\|\partial_x\phi(t)\|^2+\|\partial_x(\psi,\zeta)(t)\|_1^2\,dt\leq C(E_0+\delta^{1/5})^2.$$

Proposition 4 (ideal polytropic gas). *Let $U(t,x)$ be a solution of* (1) *in the sense stated above. Assume that $\|(\phi,\psi)(t)\|_1\leq E$, $\|\zeta(t)\|_1\leq\underline{E}$, $t\in[0,T]$, and $v(t,x)\geq m$, $\theta(t,x)\geq\underline{m}$, $(t,x)\in[0,T]\times\boldsymbol{R}$, for positive constants E, $\underline{E}$, m and $\underline{m}$. Then there are positive constants $\delta_3=\delta_3\,(E,m)$, C and m_2 not depending on T and $\gamma\in(1,\gamma_0]$ such that if $\delta=|U_+-U_-|\leq\delta_3$, then*

$$\sup_{0\leq t\leq T}\|(\phi,\psi,\zeta/\sqrt{\gamma-1})(t)\|_1^2+\int_0^T\|\partial_x\phi(t)\|^2+\|\partial_x(\psi,\zeta)(t)\|_1^2\,dt$$
$$\leq C(E_1+1)^2,\qquad \inf v(t,x)\geq m_2$$

for $\gamma\in(1,\gamma_0]$, where inf *is taken over* $[0,T]\times\boldsymbol{R}$. *The constants C and m_2 do not depend on E and m.*

These propositions are proved by the energy method employed in [6]. In particular, we use the energy function $E(U,\tilde{U})=e-\tilde{e}+\tilde{p}(v-\tilde{v})-\tilde{\theta}(s-\tilde{s})+(u-\tilde{u})^2/2$, which is reduced to $R\tilde{\theta}H(v/\tilde{v})+R\tilde{\theta}H(\theta/\tilde{\theta})/(\gamma-1)+(u-\tilde{u})^2/2$ for the case of an ideal polytropic gas, where $H(\eta)=\eta-1-\log\eta$. Though we omit the details of calculations, we remark that in our computations we have extra terms involving the derivatives $\partial_x^l\tilde{U}(t,x)$, $l=1,2$, or rapidly decreasing functions of $(t,x)\in[0,\infty)\times\boldsymbol{R}$, each of which vanishes when the basic state $\tilde{U}(t,x)$ is constant (or equivalently, $\delta=|U_+-U_-|=0$). These extra terms are estimated similarly as in [5].

References

[1] S. Kawashima and A. Matsumura: Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. Commun. Math. Phys., **101**, 97–127 (1985).

[2] P. D. Lax: Hyperbolic systems of conservation laws, II. Comm. Pure Appl. Math., **10**, 537–566 (1957).

[3] T.-P. Liu: Shock waves for compressible Navier-Stokes equations are stable (preprint).

[4] A. Matsumura and K. Nishihara: On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math., **2**, 17–25 (1985).

[5] ——: Asymptotics towards the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. ibid., **3**, 1–13 (1986).

[6] M. Okada and S. Kawashima: On the equations of one-dimensional motion of compressible viscous fluids. J. Math. Kyoto Univ., **23**, 55–71 (1983).