7. *Simple Vector Bundles over Symplectic Kähler Manifolds*

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(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 13, 1986)

1. Introduction. In a recent paper [5], Mukai has shown that the moduli space of simple sheaves on an abelian or K3 surface is smooth and has a holomorphic symplectic structure. We extend his result to higher dimensional manifolds by a differential geometric method.

A *holomorphic symplectic structure* on a complex manifold is given by a closed holomorphic 2-form ω which is non-degenerate in the sense that if $\omega(u, v)=0$ for all tangent vectors v, then $u=0$.

Let M be a compact Kähler manifold of dimension n and E a C^∞ complex vector bundle of rank r over M. Let $A^{p,q}(E)$ be the space of C^∞ (p, q)-forms over M with values in E. A *semi-connection* in E is a linear map $D'' : A^{0,0}(E) \to A^{0,1}(E)$ such that

(1.1) $$D''(as)=d''a \cdot s + aD''s$$

for all functions a on M and all sections s of E. Let $\mathcal{D}''(E)$ denote the space of semi-connections in E. Every semi-connection D'' extends uniquely to a linear map $D'' : A^{p,q}(E) \to A^{p,q+1}(E)$ such that

(1.2) $$D''(\alpha \wedge \sigma)=d''\alpha \wedge \sigma + (-1)^r \alpha \wedge D''\sigma$$

for all r-forms α on M and all $\sigma \in A^{p,q}(E)$. In particular,

(1.3) $$N(D'') := D'' \circ D'' : A^{0,0}(E) \longrightarrow A^{0,2}(E),$$

and $N(D'')$ may be considered as an element of $A^{0,2}(\mathrm{End}(E))$. A semi-connection D'' is called a *holomorphic structure* if $N(D'')=0$. Let $\mathcal{H}''(E)$ denote the set of holomorphic structures in E. If E is holomorphic, then $d'' \in \mathcal{H}''(E)$. Conversely, every $D'' \in \mathcal{H}''(E)$ comes from a unique holomorphic structure in E. The holomorphic vector bundle defined by D'' is denoted by $E^{D''}$. We call $E^{D''}$ *simple* if its endomorphisms are all of the form cI_E, where $c \in \boldsymbol{C}$. Let

(1.4) $$\mathrm{End}^0(E^{D''})=\{u \in \mathrm{End}(E^{D''})\, ;\, \mathrm{Tr}(u)=0\}.$$

Then $E^{D''}$ is simple if and only if $H^0(M, \mathrm{End}^0(E^{D''}))=0$. Let $\mathcal{S}''(E)$ denote the set of simple holomorphic structures D'' in E.

Let $GL(E)$ be the group of C^∞ automorphisms of the bundle E. Its Lie algebra $\mathfrak{gl}(E)$ is nothing but $A^{0,0}(\mathrm{End}(E))$. The group $GL(E)$ acts on $\mathcal{D}''(E)$ by

(1.5) $$D''^f = f^{-1} \circ D'' \circ f \qquad \text{for } f \in GL(E),\ D'' \in \mathcal{D}''(E).$$

Then $GL(E)$ leaves $\mathcal{H}''(E)$ and $\mathcal{S}''(E)$ invariant. With the C^∞ topology, the *moduli space* $\mathcal{S}''(E)/GL(E)$ *of simple holomorphic* structures in E is a (possibly non-Hausdorff) complex analytic space. As was shown by Kim

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*) Partially supported by NSF Grant DMS 85-02362.

[3], it is a non-singular complex manifold in a neighborhood of $[D''] \in \mathcal{S}''(E)/GL(E)$ if $H^2(M, \mathrm{End}^0(E^{D''}))=0$. This is analogous to Kodaira-Spences-Kuranishi theory of complex structures.

We are now in a position to state our result.

(1.6). **Theorem.** *Let M be a compact Kähler manifold with a holomorphic symplectic structure ω_M. Let E be a C^∞ complex vector bundle over M and let $\mathcal{S}''(E)/GL(E)$ be the moduli space of simple holomorphic vector bundles in E. Let*

$$\mathcal{M}(E)=\{[D''] \in \mathcal{S}''(E)/GL(E)\,;\, H^2(M, \mathrm{End}^0(E^{D''}))=0\}$$

so that $\mathcal{M}(E)$ is a non-singular (possibly non-Hausdorff) complex manifold. Then ω_M induces in a natural way a holomorphic symplectic structure on $\mathcal{M}(E)$.

If $\dim M=2$, then $H^2(M, \mathrm{End}(E^{D''}))$ is dual to $H^0(M, \mathrm{End}(E^{D''}))$ and hence $\mathcal{M}(E)=\mathcal{S}''(E)/GL(E)$.

2. Outline of the proof. The construction of a holomorphic symplectic structure on $\mathcal{M}(E)$ is based on the reduction theorem of Marsden-Weinstein [4]. While their theorem is proved in the differentiable case, we need its holomorphic analogue. So we recall it in the form adapted to our purpose. Let V be a complex Banach manifold with a holomorphic symplectic structure ω_V. Let G be a Banach complex Lie group acting holomorphically on V, leaving ω_V invariant. Let $\mathfrak{g}$ be the Banach Lie algebra of G and $\mathfrak{g}^*$ its dual vector space. A *moment* for the action of G is a holomorphic map $\psi : V \to \mathfrak{g}^*$ such that

(2.1) $$\langle a, d\psi_x(v)\rangle=\omega_V(a_x, v) \qquad \text{for } a \in \mathfrak{g},\ v \in T_xV,\ x \in V,$$

where $d\psi_x : T_xV \to \mathfrak{g}^*$ is the differential of ψ at x, $a_x \in T_xV$ is the vector defined by the infinitesimal action of $a \in \mathfrak{g}$, and $\langle\ ,\ \rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

(a) Assume that ψ is equivariant with respect to the coadjoint action of G, i.e.,

(2.2) $$\psi(g(x))=(\mathrm{Ad}\, g)^*(\psi(x)) \qquad \text{for } g \in G,\ x \in V.$$

Then G leaves $\psi^{-1}(0)\subset V$ invariant. The quotient space $W=\psi^{-1}(0)/G$ is called the *reduced phase space*. Let $j : \psi^{-1}(0)\to V$ be the natural injection and $\pi : \psi^{-1}(0)\to W$ the projection.

(b) Assume that $0 \in \mathfrak{g}^*$ is a weakly regular value of ψ in the sense that (i) $\psi^{-1}(0)$ is a submanifold of V and (ii) for every $x \in \psi^{-1}(0)$, the inclusion $T_x(\psi^{-1}(0))\subset \mathrm{Ker}\,(d\psi_x)$ is an equality.

(c) Assume that the action of G on $\psi^{-1}(0)$ is free and that at each point $x \in \psi^{-1}(0)$ there is a holomorphic "slice section" $S_x \subset \psi^{-1}(0)$ for the action.

Then the theorem says that under these assumptions W is a (possibly non-Hausdorff) complex manifold and there is a unique holomorphic symplectic structure ω_W on W such that $\pi^*\omega_W=j^*\omega_V$. Marsden and Weinstein assume that the action of G is proper. We assume instead only the existence of a slice at each point of $\psi^{-1}(0)$. So our manifold W may not be Hausdorff.

Now we apply the theorem to the following situation. If a semi-connection $D'' \in \mathcal{D}''(E)$ is chosen, every other element of $\mathcal{D}''(E)$ is of the form $D''+\alpha$, where $\alpha \in A^{0,1}(\mathrm{End}(E))$. So $\mathcal{D}''(E)$ is an affine space, and its tangent space at D'' can be identified with $A^{0,1}(\mathrm{End}(E))$. Taking $k > \dim M$, we consider the Sobolev space $L^2_k(D''(E))$. The action of $GL(E)$ is not effective; an element $f \in GL(E)$ acts trivially on $\mathcal{D}''(E)$ if and only if $f = cI_E$ with $c \in \boldsymbol{C}^* = \boldsymbol{C} - \{0\}$. Let

$$V = L^2_k(\mathcal{D}''(E)), \quad G = L^2_{k+1}(GL(E)/\boldsymbol{C}^*), \quad \mathfrak{g} = L^2_{k+1}(\mathfrak{gl}(E)/\boldsymbol{C}).$$

Then G acts effectively and smoothly on V. Using a holomorphic symplectic structure ω_M of M, we define a holomorphic symplectic structure ω_V on V by

$$\omega_V(\alpha, \beta) = \int_M \mathrm{Tr}(\alpha \wedge \beta) \wedge \omega_M^m \wedge \bar{\omega}_M^{m-1}, \qquad \alpha, \beta \in T_{D''}(V), \tag{2.3}$$

where α and β are considered as elements of $L^2_k(A^{0,1}(\mathrm{End}(E))) \approx T_{D''}(V)$ and $2m$ is the dimension of M. We define a moment $\psi : V \to \mathfrak{g}^*$ by

$$\langle a, \psi(D'') \rangle = -\int_M \mathrm{Tr}(a \circ N(D'')) \wedge \omega_M^m \wedge \bar{\omega}_M^{m-1}, \quad a \in \mathfrak{g}, \quad D'' \in V. \tag{2.4}$$

We verify (2.1) for ψ using the following formulas.

$$\partial_t N(D'' + t\beta)|_{t=0} = D''\beta, \qquad \text{for } \beta \in L^2_k(A^{0,1}(\mathrm{End}(E))), \tag{2.5}$$

$$\partial_t (e^{-at} \circ D'' \circ e^{at})|_{t=0} = D''a, \qquad \text{for } a \in \mathfrak{g}. \tag{2.6}$$

The latter means that $D''a$ is the tangent vector $a_{D''} \in T_{D''}(V)$ induced by the infinitesimal action of $a \in \mathfrak{g}$. Now we have

$$\begin{aligned} \langle a, d\psi_{D''}(\beta) \rangle &= -\partial_t \int_M \mathrm{Tr}(a \circ N(D'' + t\beta)) \wedge \omega_M^m \wedge \bar{\omega}_M^{m-1}|_{t=0} \\ &= -\int_M \mathrm{Tr}(a \circ D''\beta) \wedge \omega_M^m \wedge \bar{\omega}_M^{m-1} \\ &= \int_M \mathrm{Tr}(D''a \wedge \beta) \wedge \omega_M^m \wedge \bar{\omega}_M^{m-1} \\ &= \omega_V(D''a, \beta) = \omega_V(a_{D''}, \beta). \end{aligned} \tag{2.7}$$

This verifies (2.1) for ψ. From $\mathrm{Tr}(a \circ N(D''^f)) = \mathrm{Tr}(a \circ f^{-1} \circ N(D'') \circ f) = \mathrm{Tr}(faf^{-1} \circ N(D''))$, we obtain

$$\langle a, \psi(D''^f) \rangle = \langle faf^{-1}, \psi(D'') \rangle, \tag{2.8}$$

showing that ψ is coad(G)-equivariant.

To verify (b) we have to take a certain open subset V' of V. Let $D'' \in \psi^{-1}(0) = \{D'' \in V;\ N(D'') = 0\}$. If $\langle a, d\psi_{D''}(\beta) \rangle = 0$ for all $\beta \in T_{D''}(V)$, then (2.7) implies $D''a = 0$. So we consider the open subset V' of V consisting of D'' such that $a = 0$ is the only solution of $D''a = 0$ in $\mathfrak{g} = L^2_{k+1}(A^{0,0}(\mathrm{End}^0(E)))$. Then $0 \in \mathfrak{g}^*$ is a weakly regular value of $\psi|_{V'}$ and

$$\psi^{-1}(0) \cap V' = \{D'' \in V;\ N(D'') = 0 \text{ and } E^{D''} \text{ is simple}\}. \tag{2.9}$$

Let $f \in G$ and $D'' \in V'$. If $D''^f = D''$, i.e., $D'' \circ f = f \circ D''$, then $D''f = 0$ and hence $f = cI_E$ with $c \in \boldsymbol{C}^*$, showing that G acts freely on V.

Finally, we define a slice $S_{D''}$ through D'' by

$$S_{D''} = \{D'' + \alpha \in V;\ D''\alpha + \alpha \wedge \alpha = 0 \text{ and } D''^*\alpha = 0\}, \tag{2.10}$$

where D''^* is the adjoint of D''. Then in a neighborhood of D'' for which $H^2(M, \mathrm{End}^0(E^{D''})) = 0$, the slice $S_{D''}$ is a non-singular complex submanifold of V.

Remark. In Atiyah-Bott [1], the original Marsden-Weinstein theorem for the real case is used to construct a real symplectic structure or Kähler form on the moduli space of stable bundles over a curve. In [2] Itoh constructs also a Kähler form on the moduli space of anti-self-dual connections on a compact Kähler surface using slices.

The Kähler metric of M induces a Kähler metric on the non-singular part of the moduli space $\hat{\mathcal{M}}(E)$ ($\subset \mathcal{M}(E)$) of irreducible Einstein-Hermitian connections. If the metric on M is Ricci-flat so that the symplectic form ω_M is parallel, so is the induced Kähler metric on $\hat{\mathcal{M}}(E)$.

References

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