41. *Homology of a Local System on the Complement of Hyperplanes*

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1. Introduction and statement of results. Let $\{A_j\}_{1\leqslant j\leqslant m}$ be a finite family of complex affine hyperplanes in $\boldsymbol{C}^n$ and let $\mathcal{L}$ be a local system on the complement $X=\boldsymbol{C}^n-\bigcup_{j=1}^m A_j$. The vanishing of homology $H_j(X,\mathcal{L})$, $j\neq n$, for a "generic" local system $\mathcal{L}$ was treated by K. Aomoto [2] and M. Kita-M. Noumi [7] from different points of view. The object of this note is to give a simple criterion for such vanishing of homology and to give a basis of $H_n(X,\mathcal{L})$. We denote by f_j, $1\leqslant j\leqslant m$, a linear form with $\operatorname{Ker} f_j=A_j$ and let A_{m+1} denote the hyperplane at infinity. We consider a regular connection of the form $\Omega=\sum_{j=1}^m P_j d\log f_j$, $P_j\in\operatorname{End}(V)$, where V is a finite dimensional complex vector space. Let us observe that the connection Ω is integrable if and only if $[P_{j_\nu}, P_{j_1}+\cdots+P_{j_q}]=0$, $1\leqslant\nu\leqslant q$, for any maximal family $\{A_{j_\nu}\}_{1\leqslant\nu\leqslant q}$ such that $\operatorname{codim}_{\boldsymbol{C}}[A_{j_1}\cap\cdots\cap A_{j_q}]=2$ (see [1]). These relations are related to the lower central series of the fundamental group of X (see [8], [9]). Let P_{m+1} denote the residue along A_{m+1}. We have $P_1+\cdots+P_{m+1}=0$.

Let us suppose that Ω is integrable in the followings. The connection Ω is said to be *generic with respect to the hyperplanes* $\{A_j\}_{1\leqslant j\leqslant m+1}$ if the following conditions are satisfied:

(1.1) (i) Any eigenvalue of P_j, $1\leqslant j\leqslant m+1$, is not an integer.

(ii) For any maximal subfamily $\{A_{j_\nu}\}_{1\leqslant\nu\leqslant q}$, such that $\operatorname{codim}_{\boldsymbol{C}}[A_{j_1}\cap\cdots\cap A_{j_q}]=r$ with some $r<q$, any eigenvalue of $P_{j_1}+\cdots+P_{j_q}$ is not an integer.

The solutions of the system of differential equations $dY+\Omega.Y=0$ defines a local system $\mathcal{L}$ on X, which determines a homomorphism $\rho:\pi_1(X,x_0)\to\operatorname{Aut}(\mathcal{L}_{x_0})$. Let $\tilde{X}$ be the universal covering of X. The homology $H_j(X,\mathcal{L})$ is defined to be the j-th homology of the complex $C.(\tilde{X})\otimes_{\boldsymbol{Z}[G]}\mathcal{L}_{x_0}$, where $G=\pi_1(X,x_0)$ and the space of chains of $\tilde{X}$ is considered as a right $\boldsymbol{Z}[G]$-module via covering transformations and $\mathcal{L}_{x_0}$ is a left $\boldsymbol{Z}[G]$-module via ρ. The homology of the locally finite (possibly infinite) chains is defined in the same way and we denote it by $H_j^{lf}(X,\mathcal{L})$.

Theorem 1. *Let us suppose that the integrable connection* $\Omega=\sum_{j=1}^m P_j \times d\log f_j$ *is generic with respect to the hyperplanes* $\{A_j\}_{1\leqslant j\leqslant m+1}$ *in the sense of* (1.1). *Let* $\mathcal{L}$ *denote the local system over* $X=\boldsymbol{C}^n-\bigcup_{j=1}^m A_j$ *associated with* Ω. *Then we have an isomorphism*

$$H_j^{lf}(X,\mathcal{L})\cong H_j(X,\mathcal{L})\qquad \textit{for any } j, \tag{1.2}$$

and we have

(1.3) $$H_j(X, \mathcal{L})=0 \qquad \text{if } j\neq n.$$

For $0\leqslant t\leqslant 1$, we consider the 1-parameter family of hyperplanes $A_{j,t}$, $1\leqslant j\leqslant m$, defined by $f_j-(1-t)c_j$, where $c_j=f_j(0)$. We put $A_{m+1,t}=A_{m+1}$. Let us define Ω_t by $\sum_{j=1}^m P_j d\log(f_j-(1-t)c_j)$ for a fixed t. We observe that the connection Ω_0 is integrable.

Theorem 2. *Let us suppose that the hyperplanes $\{A_j\}_{1\leqslant j\leqslant m+1}$ and the connection Ω_0 defined above satisfy the following conditions:*

(i) *A_j, $1\leqslant j\leqslant m$, is defined by a real linear form.*

(ii) *The union $\bigcup_{j=1}^m A_j$ intersects transversely with the hyperplane at infinity A_{m+1}.*

(iii) *Ω_0 is generic with respect to the hyperplanes $\{A_{j,0}\}_{1\leqslant j\leqslant m+1}$ in the sense of* (1.1).

Let $\{\Delta_i\}_{1\leqslant i\leqslant k}$ denote the relatively compact chambers of $\{\boldsymbol{C}^n-\bigcup_{j=1}^m A_j\}\cap \boldsymbol{R}^n$ and let $\mathcal{L}$ denote the local system on $X=\boldsymbol{C}^n-\bigcup_{j=1}^m A_j$ associated with Ω. Then we have

(1.4) $$H_j^{lf}(X, \mathcal{L})=0 \qquad \text{if } j\neq n.$$

(1.5) $$H_j(X, \mathcal{L})=0 \qquad \text{if } j\neq n,$$

and we have a decomposition

(1.6) $$H_n^{lf}(X, \mathcal{L})\cong \bigoplus_{i=1}^{k}([\Delta_i]\otimes\mathcal{L}_{x_0}).$$

The method developed in this note is in some sense a generalization of Section 2 of [5], where the cohomology of a rank one local system on a punctured projective line is explained. I would like to thank Prof. K. Aomoto and Prof. E. Viehweg for valuable discussions concerning the subject of this note.

2. Proof of Theorem 1. Let $\mu:\widehat{CP^n}\to CP^n$ be the composition of the monoidal transforms along $A_{j_1}\cap\cdots\cap A_{j_q}$ for each maximal subfamily $\{A_{j_\nu}\}_{1\leqslant\nu\leqslant q}$ such that $\operatorname{codim}_C[A_{j_1}\cap\cdots\cap A_{j_q}]=r$ with some $r<q$. Then $\mu^{-1}(\bigcup_{j=1}^{m+1}A_j)$ is a divisor with normal crossings and the residue of Ω along the exceptional divisor $\mu^{-1}(A_{j_1}\cap\cdots\cap A_{j_q})$ is given by $P_{j_1}+\cdots+P_{j_q}$. The eigenvalues of $\rho(\gamma)$ for a normal loop γ around the proper transform of A_j and around $\mu^{-1}(A_{j_1}\cap\cdots\cap A_{j_q})$ are the same as the eigenvalues of $\exp 2\pi\sqrt{-1}P_j$ and of $\exp 2\pi\sqrt{-1}(P_{j_1}+\cdots+P_{j_q})$, respectively (see [4]). Hence Theorem 1 is a consequence of the following Proposition.

(2.1) **Proposition.** *Let X be a smooth affine variety over $\boldsymbol{C}$ of dimension n. Let V be its smooth compactification such that $D=V-X$ is a divisor with normal crossings. Let $\mathcal{L}$ be a local system on X and let $\rho:\pi_1(X, x_0)\to\operatorname{Aut}(\mathcal{L}_{x_0})$ be its associated monodromy. If 1 is not an eigenvalue of $\rho(\gamma_j)$ for a normal loop γ_j around any irreducible component D_j of D, then we have*

$$H_j^{lf}(X, \mathcal{L})\cong H_j(X, \mathcal{L}) \quad \text{for any } j, \quad H_j(X, \mathcal{L})=0 \quad \text{if } j\neq n.$$

Proof of Proposition. Let $i: X\to V$ denote the inclusion. By the hypothesis of Proposition we have $i_*\mathcal{L}=i_!\mathcal{L}$, where $i_!\mathcal{L}$ stands for the extension of $\mathcal{L}$ by 0. We have the following isomorphisms:

(2.2) $$H^j(V, i_*\mathcal{L}) \cong H^j(X, \mathcal{L}), \qquad H^j(V, i_!\mathcal{L}) \cong H^j_c(X, \mathcal{L}),$$

where H^j_c is the cohomology with compact support. On the other hand, we have the following perfect pairings by the Poincaré duality:

(2.3) $$H^j(X, \mathcal{L}) \otimes H^{lf}_{2n-j}(X, \mathcal{L}) \longrightarrow \boldsymbol{C}, \qquad H^j_c(X, \mathcal{L}) \otimes H_{2n-j}(X, \mathcal{L}) \longrightarrow \boldsymbol{C}.$$

Hence we obtain the first assertion. By means of the Lefschetz theorem, we have

(2.4) $$H^j(X, \mathcal{L}) = 0 \qquad \text{for } j > n.$$

By combining this with (2.3), we obtain

(2.5) $$H^{lf}_j(X, \mathcal{L}) = 0 \qquad \text{for } j < n.$$

The second assertion of Proposition follows from (2.4) and (2.5) together with the first assertion.

3. Proof of Theorem 2. Let D_i, $1 \leqslant i \leqslant k$, be an open ball in $\boldsymbol{R}^n$ and we fix a homeomorphism $\alpha_i : D_i \to \Delta_i$, which induces an injective map $\alpha : \coprod_{i=1}^k D_i \to X$. We have a long exact sequence:

(3.1) $$\longrightarrow H^{lf}_j(\coprod D_i, \alpha^*\mathcal{L}) \longrightarrow H^{lf}_j(X, \mathcal{L}) \longrightarrow H^{lf}_j(X - \coprod \Delta_i, \mathcal{L}) \longrightarrow.$$

Since $\alpha^*\mathcal{L}$ is a trivial local system, we have $H^{lf}_j(\coprod D_i, \alpha^*\mathcal{L}) = 0$ if $j \neq n$ and we have a decomposition $H^{lf}_n(\coprod D_i, \alpha^*\mathcal{L}) = \oplus_{i=1}^k [D_i] \otimes (\alpha^*\mathcal{L})_z$ with some $z \in \coprod D_i$. It remains to show that $H^{lf}_j(X - \coprod \Delta_i, \mathcal{L}) = 0$ for any j. By the Poincaré duality we have $H^{lf}_j(X - \coprod \Delta_i, \mathcal{L}) \cong H^{2n-j}(X - \coprod \Delta_i, \mathcal{L})^\vee$. Since $X - \coprod \Delta_i$ is homotopy equivalent to $\boldsymbol{C}^n - \cup_{j=1}^m A_{j,0}$, we get an isomorphism

$$H^{2n-j}(X - \coprod \Delta_i, \mathcal{L}) \cong H^{2n-j}(\boldsymbol{C}^n - \bigcup_{j=1}^m A_{j,0}, \mathcal{L}'),$$

where $\mathcal{L}'$ denotes the local system associated with Ω_0. By applying Theorem 1 to the dual local system of $\mathcal{L}'$, we have $H^{lf}_j(X - \coprod \Delta_i, \mathcal{L}) = 0$ if $j \neq n$. On the other hand the Euler-Poincaré characteristic of $\boldsymbol{C}^n - \cup_{j=1}^m A_{j,0}$ is zero (see for example [10]). Thus we have proved the assertions (1.4) and (1.6). The assertion (1.5) follows from the Poincaré duality and (1.4). This completes the proof of Theorem 2. Consequently we have shown that the number of the relatively compact chambers of $X \cap \boldsymbol{R}^n$ is equal to the Euler-Poincaré characteristic of X up to sign (cf. [10]).

4. Examples. (4.1) Let $\{A_j\}_{1 \leqslant j \leqslant m}$ be a family of affine hyperplanes in general position in $\boldsymbol{C}^n$. Then the connection $\Omega = \sum_{j=1}^m P_j d \log f_j$ satisfies the hypothesis of Theorem 2 if any eigenvalue of P_j, $1 \leqslant j \leqslant m$, and $\sum_{j=1}^m P_j$ is not an integer (cf. [6], [7]).

(4.2) Let us suppose that there exists a linear map $\pi : X = \boldsymbol{C}^n - \cup_{j=1}^m A_j \to \boldsymbol{C}^p$ such that the image of π and the fiber of π are the complements of hyperplanes. Let Z denote a fiber and let $i : Z \to X$ be the inclusion map. If the hypothesis of Theorem 2 is satisfied for $i^*\mathcal{L}$ and $\boldsymbol{R}^{n-p}\pi_*\mathcal{L}$, then by using Leray spectral sequence we obtain (1.4)–(1.6) for $\mathcal{L}$ over X.

(4.3) We put $M_n = \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n ; z_i \neq z_j \text{ if } i \neq j\}$. Let $\mathcal{L}$ be a local system over M_n defined by associating $\exp 2\pi\sqrt{-1}\,\mu_{ij}$ to a normal loop around $z_i = z_j$. Let X be the fiber over $(0, 1)$ of a natural projection $M_4 \to M_2$. By applying (4.2) for $\mathcal{L}$ on X, we obtain (1.4)–(1.6) if $\mu_{14}+\mu_{24}+\mu_{34}\notin \boldsymbol{Z}$, $\mu_{13}+\mu_{23}+\mu_{14}+\mu_{24}+\mu_{34}\notin \boldsymbol{Z}$ (cf. [3]).

References

[1] Aomoto, K.: Une remarque sur la solution des équations de *L*. Schlesinger et Lappo-Danilevski. J. Fac. Sci. Univ. Tokyo, **17**, 341–354 (1970).

[2] ——: On vanishing of cohomology attached to certain many valued meromorphic functions. J. Math. Soc. Japan, **27**, 248–255 (1975).

[3] ——: Gauss-Manin connection of integral of difference products (preprint, 1985).

[4] Deligne, P.: Equations différentielles à points singuliers réguliers. Lect. Notes in Math., Springer, vol. 163 (1970).

[5] Deligne, P. and Mostow, G. D.: Monodromy of hypergeometric functions and non-lattice integral monodromy (preprint, IHES, 1983).

[6] Hattori, A.: Topology of $\boldsymbol{C}^n$ minus a finite number of affine hyperplanes in general position. J. Fac. Sci. Univ. Tokyo, **22**, 205–219 (1975).

[7] Kita, M. and Noumi, M.: On the structure of cohomology groups attached to certain many valued analytic functions. Japan J. Math., **9**, 113–157 (1983).

[8] Kohno, T.: On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces. Nagoya Math. J., **92**, 21–37 (1983).

[9] ——: Série de Poincaré-Koszul associée aux groupes de tresses pures. Invent. Math., **82**, 57–75 (1985).

[10] Zaslavski, T.: Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes. Mem. Amer. Math. Soc., **154**, 1–102 (1975).