40. *Multi-Tensors of Differential Forms on the Siegel Modular Variety and on its Subvarieties*

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Introduction. Let $A_n=H_n/\Gamma_n$, where H_n is the Siegel space $\{Z\in M_n(\boldsymbol{C})\,|\,{}^tZ=Z,\ \mathrm{Im}\,Z>0\}$, and $\Gamma_n=Sp_{2n}(\boldsymbol{Z})$. A_n is shown to be of general type for $n\geqq 9$ by Tai [5] ($n=8$ by Freitag [2], $n=7$ by Mumford [4]). Subvarieties of A_n are expected to have the same property if they are not too special. We have the following theorem. The details of the proof are included in Tsuyumine [9].

Theorem. *Let $n\geqq 10$. Then any subvariety in A_n of codimension one is of general type.*

We have the following corollary to this theorem (cf. Freitag [3]). We denote by $\Gamma_n(l)$ the principal congruence subgroup of level l, and by $A_{n,l}$ the quotient space $H_n/\Gamma_n(l)$.

Corollary. *Let $n\geqq 10$. Then the birational automorphism group of $A_{n,l}$ equals* $\mathrm{Aut}(A_{n,l})\simeq\Gamma_n/\pm\Gamma_n(l)$. *In particular, A_n has no non-trivial birational automorphism.*

§1. **Preliminaries.** The symplectic group $Sp_{2n}(\boldsymbol{R})$ acts on H_n by the usual symplectic substitution:

$$Z\longrightarrow MZ=(AZ+B)(CZ+D)^{-1},$$
$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp_{2n}(\boldsymbol{R}).$$

Let $Z=(z_{ij})$, and let

$$\omega_{ij}=(-1)^{i+j}e_{ij}dz_{11}\wedge dz_{12}\wedge\cdots\wedge\check{dz_{ij}}\wedge\cdots\wedge dz_{nn},\qquad e_{ij}=\begin{cases}1 & i\neq j,\\ 2 & i=j,\end{cases}$$

for $1\leqq i\leqq j\leqq n$. Let $\omega=(\omega_{ij})$. Then we have

$$M\cdot\omega=|CZ+D|^{-n-1}(CZ+D)\omega^t(CZ+D),$$

and so

$$M\cdot\omega^{\otimes r}=|CZ+D|^{-r(n+1)}(CZ+D)^{\otimes r}\omega^{\otimes r\,t}(CZ+D)^{\otimes r}.$$

A Siegel modular form f admits the Fourier expansion $f(Z)=\sum_{S\geqq 0}a(S)\boldsymbol{e}(\mathrm{tr}\,((1/2)SZ))$, $\boldsymbol{e}(\ \ )$ standing for $\exp(2\pi\sqrt{-1}\ \ )$. f is said to vanish to order α (at the cusp) if α is the minimum integer such that $a(S)=0$ for S with $\min_{g\in Z^n,\neq 0}\{(1/2)S[g]\}<\alpha$, $S[g]$ denoting ${}^tgSg$. We denote it by $\mathrm{ord}\,(f)$.

§2. **Theta series.** Let m be an integer with $m\geqq 2(n-1)$, and let η be a complex $m\times(n-1)$ matrix satisfying both ${}^t\eta\eta=0$ and rank $\eta=n-1$. η_i $(1\leqq i\leqq n)$ denotes an $(n-1)\times n$ matrix given by

$$\eta_i=\begin{bmatrix}1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{bmatrix}.$$

We fix a positive symmetric matrix F of size m with rational coefficients. Let r be a positive integer, and let I, J be ordered collections of r integers in $\{1, \cdots, n\}$ where a repeated choice is allowed. We define a theta series associated with F by setting

$$\theta_F^{(I,J)}\begin{bmatrix}u\\v\end{bmatrix}(Z)=\operatorname{sgn}(I)\operatorname{sgn}(J)\sum_G\prod_{i\in I}|\eta_i\,{}^t(G+u)F^{1/2}\eta|\prod_{j\in J}|\eta_j\,{}^t(G+u)F^{1/2}\eta|$$
$$\times\boldsymbol{e}\left(\operatorname{tr}\left(\frac{1}{2}ZF[G+u]+{}^t(G+u)v\right)\right)$$

where G runs through all $m\times n$ integral matrices, and u, v are m, n matrices with rational coefficients. We define $\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z)$ to be a square matrix of size n^r whose (k, l)-entry is $\theta_F^{(I,J)}\begin{bmatrix}u\\v\end{bmatrix}(Z)$ where $k=1+\sum_{s=1}^r(i_s-1)n^{s-1}$, $l=1+\sum_{s=1}^r(j_s-1)n^{s-1}$ with $I=\{i_1, \cdots, i_r\}$, $J=\{j_1, \cdots, j_r\}$.

Proposition 1. *There is an integer l such that*

$$\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(MZ)=\chi(M)|CZ+D|^{(m/2)+2r}({}^t(CZ+D)^{-1})^{\otimes r}\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(Z)((CZ+D)^{-1})^{\otimes r}$$

holds for any $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_n(l)$ where χ is a map of $\Gamma_n(l)$ to the set of roots of unity. χ is killed by some power.

The proof is done by the similar method as in Andrianov and Maloletkin [1], Tsuyumine [6], [7].

§3. Multi-tensors of differentials. Let r' be a positive integer such that $\chi^{r'}=1$. Let $\{M_j\}$ be any system of representatives of $\Gamma_n \bmod \Gamma_n(l)$. Let us put

$$\Psi(Z)=\sum_j|C_jZ+D_j|^{-((m/2)+2r)r'}\,{}^t(C_jZ+D_j)^{\otimes rr'}\left(\Psi_{F,r}\begin{bmatrix}u\\v\end{bmatrix}(M_jZ)\right)^{\otimes r'}(C_jZ+D_j)^{\otimes rr'}$$

where $M_j=\begin{pmatrix}A_j & B_j\\ C_j & D_j\end{pmatrix}$. Then $\Psi(Z)$ satisfies

$$(*)\qquad \Psi(Z)=|CZ+D|^{((m/2)+2r)r'}({}^t(CZ+D)^{-1})^{\otimes rr'}\Psi(Z)((CZ+D)^{-1})^{\otimes rr'}$$

for $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_n$.

The following is shown by calculation:

Proposition 2. *Let Z_0 be any point of H_n, and let W be any nonzero complex symmetric matrix of size n. Let m be an integer with $m\geqq 2(n-1)$. Then for infinitely many r and for infinitely many r', there is a symmetric matrix $\Psi(Z)$ of size $n^{rr'}$ satisfying the above $(*)$ for Γ_n such that* $\operatorname{tr}(\Psi(Z_0)W^{\otimes rr'})\neq 0$.

Let us put

$$\lambda_{m,r,r'}=\mathrm{tr}\,(\Psi(Z)\omega^{\otimes rr'}).$$

By (*) and by the transformation formula of $\omega^{\otimes rr'}$, we have the following:

Proposition 3. *Suppose $r(n-1)\geqq m/2$. Then for any modular form f of weight $(r(n-1)-(m/2))r'$, $f\lambda_{m,r,r'}$ is a Γ_n-invariant form in $(\Omega_{H_n}^{N-1})^{\otimes rr'}$, $N=n(n+1)/2$.*

Let A_n° denote the smooth locus of A_n. If $n\geqq 3$, then A_n° is the complement of the image of the fixed point set by the canonical projection $\pi: H_n \to A_n$. So $f\lambda_{m,r,r'}$ in Proposition 3 can be regarded as a section of $(\Omega_{A_n^{\circ}}^{N-1})^{\otimes rr'}$ if $n\geqq 3$. By the similar argument as in Tai [5], the extendability of $f\lambda_{m,r,r'}$ to a projective nonsingular model of A_n can be discussed.

Proposition 4. *Let $n\geqq 7$. If f is a modular form of weight $(r(n-1)-(m/2))r'$ with $\mathrm{ord}\,(f)\geqq rr'$, then a multi-tensor $f\lambda_{m,r,r'}$ of differentials extends holomorphically to a projective nonsingular model of A_n.*

There are many modular forms satisfying the condition in Proposition 4, provided that $n\geqq 10$ (cf. Freitag [3]). Indeed for a fixed subvariety D of codimension one, there are lots of such modular forms f such that $f\not\equiv 0$ on D. The restriction of $f\lambda_{m,r,r'}$ to D gives a pluri-canonical differential form on it. So, our theorem is derived from the following lemma, which is a consequence of Proposition 2 where the key is that a subvariety in A_n of codimension one is defined by a single modular form if $n\geqq 3$ (cf. Tsuyumine [8]).

Lemma. *Let $n\geqq 3$. Let D be any subvariety in A_n of codimension one. Then for infinitely many r and for infinitely many r' there are $\lambda_{m,r,r'}$ whose restrictions to $\pi^{-1}(D)$ do not vanish identically.*

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