34. On a Local Existence Theorem for Quasilinear Hyperbolic Mixed Problems with Neumann Type Boundary Conditions

By Gen NAKAMURA*) and Yoshihiro SHIBATA**)

(Communicated by Kôsaku YOSIDA, M. J. A., April 14, 1986)

§1. **Introduction.** Let S be a C^∞ and compact hypersurface in $\boldsymbol{R}^n$ and Ω be the interior or exterior domain of S. We shall consider the local existence in time of classical solutions for the following Neumann problem:

$$
\text{(N.P)}\quad
\begin{cases}
\underset{\sim}{P}(\underset{\sim}{u})_a=\partial_t^2 u_a-\sum_{i=1}^n \partial_i(A_{ia}(t,x,\bar{D}_x^1\underset{\sim}{u}))+\Phi_a(t,x,\bar{D}^1\underset{\sim}{u}) \\
\qquad =f_a(t,x) \quad \text{in } [0,T]\times\Omega, \\
\underset{\sim}{Q}(\underset{\sim}{u})_a=\sum_{i=1}^n \nu_i(x)A_{ia}(t,x,\bar{D}_x^1\underset{\sim}{u})+\Psi_a(t,x,\underset{\sim}{u})=g_a(t,x) \quad \text{on } [0,T]\times S, \\
u_a(0,x)=u_a^0(x), \quad (\partial_t u_a)(0,x)=u_a^1(x) \quad \text{in } \Omega
\end{cases}
$$

for $a=1,\cdots,m$. Here, $\underset{\sim}{u}=(u_1,\cdots,u_m)$, $\bar{D}_x^1\underset{\sim}{u}=(\partial_i u_a;\ i=1,\cdots,n,\ a=1,\cdots,m,\ u_i;\ i=1,\cdots,m)$, $\bar{D}^1\underset{\sim}{u}=(\partial_t u_a;\ a=1,\cdots,m,\ \bar{D}_x^1\underset{\sim}{u})$, $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$ and $\nu(x)=(\nu_1(x),\cdots,\nu_n(x))$ is the outer unit normal of S at $x\in S$. We also try to obtain a sharp energy estimate of the regularity of solution in terms of the data and the operators $\underset{\sim}{P}$, $\underset{\sim}{Q}$.

Our result has applications to the classical nonlinear wave equation with Neumann or third kind boundary condition and the equation of motion describing the small deformation of a homogeneous, isotropic, hyperelastic material under action of gravity and surface force of dead load type.

Although there are many works for the Cauchy problem and the Dirichlet problem (see [1]~[6]), it seems that the lack of the good estimate for the linearized Neumann problem such as for the Cauchy problem or the Dirichlet problem has kept away from proving the local existence theorem for the Neumann problem. The deficiency of the estimate is the derivative loss which breaks down the usual iteration process. Here, in order to avoid any misunderstanding, we add a comment. Namely, if a rough estimate for the regularity of the solution is enough and if we restrict to the case $m=1$, a global existence theorem is proved in [7] by using the Nash-Moser technique.

The idea of the proof is to introduce the new unknown $\partial_t\underset{\sim}{u}$ and replace (N.P) by the Neumann problem for some equivalent hyperbolic-elliptic system with respect to the unknowns $(\underset{\sim}{u}, \partial_t\underset{\sim}{u})$. Then we can get an estimate which is good enough to carry out the usual iteration process for the new Neumann problem.

§2. **Result and examples.** Before stating our main result, we list

*) Department of Mathematics, Josai University.
**) Department of Mathematics, University of Tsukuba.

our notations. For any multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_m)$, a function f, a vector valued function $\underset{\sim}{g}=(g_1, \cdots, g_m)$ and $L \in \boldsymbol{Z}_+$ ($\boldsymbol{Z}_+$ being the set of all non-negative integers), we put

$$D^L f=(\partial_t^j \partial_x^\alpha f;\ j+|\alpha|=L), \qquad \bar{D}^L f=(\partial_t^j \partial_x^\alpha f;\ j+|\alpha|\leq L),$$
$$D_x^L f=(\partial_x^\alpha f;\ |\alpha|=L), \qquad \bar{D}_x^L f=(\partial_x^\alpha f;\ |\alpha|\leq L),$$
$$\underset{\sim}{g}^\beta=g_1^{\beta_1}\cdots g_m^{\beta_m}, \qquad D_x^L \underset{\sim}{g}=(D_x^L g_1, \cdots, D_x^L g_m),$$

and so on. L^2 and $\|\ \|$ denote the usual L^2 function space defined in Ω and its norm, respectively. For any $L \in \boldsymbol{Z}_+$ we put

$$H^L=\{u(\text{or } \underset{\sim}{u}) \in L^2;\ \|u\|_L \ (\text{or } \|\underset{\sim}{u}\|_L)=\|\bar{D}_x^L u\| \ (\text{or } \|\bar{D}_x^L \underset{\sim}{u}\|)<\infty\}$$

and $B^L(\omega)$ (ω being an open set) denotes the set

$$\{u \in C^L(\bar{\omega});\ |u|_{B^L(\bar{\omega})}=\textstyle\sum_{|\alpha|\leq L} \sup_\Omega |\partial_x^\alpha u|<\infty\}.$$

In particular we put $\|u\|_\infty=\sup_\Omega |u(x)|$.

For any $k \in \boldsymbol{R}$, put $\ll u \gg_k=\int_S |(1-\Delta_s)^{k/2} u|^2 dS$, $\ll u \gg_0=\ll u \gg$ where Δ_s is the Laplace-Beltrami operator on S. Then $H^k(S)$ is defined by $H^k(S) =\{u;\ \ll u \gg_k<\infty\}$.

For $T>0$, $L \in \boldsymbol{Z}_+$ and a Banach space E, $C^L([0, T]; E)$ denotes the set of all E-valued functions having all the derivatives of order $\leq L$ continuous on $[0, T]$. Furthermore, put

$$E_T^{L,k}=\textstyle\bigcap_{j=0}^L C^j([0, T]; H^{L+k-j}), \qquad E_T^{L,0}=E_T^L,$$
$$S_T^{L,k}=\textstyle\bigcap_{j=0}^L C^j([0, T]; H^{L+k-j}(S)), \qquad |u|_{L,k,T}=\sup_{0\leq t\leq T} \|\bar{D}^L \bar{D}_x^k u(t, \cdot)\|,$$
$$\langle u \rangle_{L,k,T}=\sup_{0\leq t\leq T} \ll \bar{D}^L u(t, \cdot) \gg_k, \qquad |u|_{L,0,T}=|u|_{L,T}.$$

We also use the same notation for the vector $\underset{\sim}{u}$.

For positive integers s, i, a function $H=H(t, x, \mu)$ ($\mu=(\mu_1, \cdots, \mu_s)$), vectors $\underset{\sim}{u}=(u_1, \cdots, u_s)$, $\underset{\sim}{v}_j=(v_j^1, \cdots, v_j^s)$, we put

$$(d^i H)(t, x, \underset{\sim}{u})(v_1, \cdots, v_i)=(\partial^i H/\partial \eta_1 \cdots \partial \eta_i)(t, x, \underset{\sim}{u}+\textstyle\sum_{j=1}^i \eta_j \underset{\sim}{v}_j)$$

for $\eta_1=\cdots=\eta_i=0$. Moreover, put

$$\underset{\sim}{U}'=(u_{i,a};\ i=1, \cdots, n,\ a=1, \cdots, m,\ u_a;\ a=1, \cdots, m),$$
$$\underset{\sim}{U}=(u_{0,a};\ a=1, \cdots, m;\ \underset{\sim}{U}'), \qquad \underset{\sim}{u}=(u_1, \cdots, u_m),$$
$$u_{i,a}=\partial u_a/\partial x_i, \qquad u_{0,a}=\partial u_a/\partial t$$

and let $A_{ia}(t, x, \underset{\sim}{U}')$, $\Phi_a(t, x, \underset{\sim}{U})$, $\Psi_a(t, x, \underset{\sim}{u})$ be real valued C^∞ functions defined on $|\underset{\sim}{U}'|\leq 3U_0$, $|\underset{\sim}{U}|\leq 3U_0$, $|\underset{\sim}{u}|\leq 3U_0$, $t \in [0, T_0]$, $x \in \bar{\Omega}$ for fixed positive constants T_0, U_0. Then, put

$$\underset{\sim}{A}_D=\textstyle\sum_{a=1}^m \sum_{j+|\alpha|+k\leq D}[\sum_{i=1}^n \sup_1 |\partial_t^j \partial_x^\alpha d^k A_{ia}(t, x, \underset{\sim}{U}')|$$
$$+\sup_2 |\partial_t^j \partial_x^\alpha d^k \Phi_a(t, x, U)|+\sup_3 |\partial_t^j \partial_x^\alpha d^k \Psi_a(t, x, \underset{\sim}{u})|],$$
$$A_{iajb}=\partial A_{ia}/\partial u_{j,b}$$

where $\sup_1$, $\sup_2$, $\sup_3$ are taken over the sets

$$E_1=[0, T_0]\times\bar{\Omega}\times\{|\underset{\sim}{U}'|\leq 3U_0\}, \qquad E_2=[0, T_0]\times\bar{\Omega}\times\{|\underset{\sim}{U}|\leq 3U_0\},$$
$$E_3=[0, T_0]\times\bar{\Omega}\times\{|\underset{\sim}{u}|\leq 3U_0\}$$

respectively.

Now we assume the following assumptions (A-1)～(A-3).

(A-1) $A_{ia}(t, x, 0)=\Phi_a(t, x, 0)=\Psi_a(t, x, 0)=0$ $(i=1, \cdots, n;\ a=1, \cdots, m)$.

(A-2) $A_{iajb}(t, x, \underset{\sim}{U}')=A_{jbia}(t, x, \underset{\sim}{U}')$ on $E_1 (a, b=1, \cdots, m;\ i, j=1, \cdots, n)$.

(A-3) There exist positive constants δ, d such that, for any $t\in[0, T_0]$, $\underset{\sim}{v}\in H^1(\Omega)$ and $\underset{\sim}{U}'\in L^\infty(\Omega)$ satisfying

$$\|\underset{\sim}{U}'\|_\infty \leq 3U_0,$$

$$\sum_{i,j=1}^n \sum_{a,b=1}^m \int_\Omega A_{iajb}(t, x, \underset{\sim}{U}'(x))\partial_j v_b(x)\overline{\partial_i v_a(x)}dx \geq \delta\|\underset{\sim}{v}\|_1^2 - d\|\underset{\sim}{v}\|^2.$$

Theorem. *Under the above assumptions* (A-1)$\sim$(A-3), *let* $L\geq[n/2]+8$ *be an integer and* $\underset{\sim}{u}^0\in H^L(\Omega)$, $\underset{\sim}{u}^1\in H^{L-1}(\Omega)$, $\underset{\sim}{f}\in E_T^{L-1}$, $\underset{\sim}{g}\in S_T^{L-1,1/2}$ *be the data of the Neumann problem* (N.P) *which satisfy* $\|\bar{D}_x^1\underset{\sim}{u}^0\|_\infty+\|\underset{\sim}{u}^1\|_\infty\leq U_0$ *and the* $L-2$*-th order compatibility condition (defined in Remark after this theorem). Then, if* $\|\underset{\sim}{u}^0\|_{[n/2]+8}+\|\underset{\sim}{u}^1\|_{[n/2]+7}+|\underset{\sim}{f}|_{[n/2]+7,T_0}+\langle \underset{\sim}{g}\rangle_{[n/2]+6,1/2,T_0}\leq B$ *and taking* D *appropriately, there exists* $T(0<T\leq T_0)$ *depending only on* n, m, Ω, T_0, $\underset{\sim}{A}_D$, δ, d *and* B *such that the Neumann problem* (N.P) *admits a unique solution* $\underset{\sim}{u}\in E_T^L$ *satisfying the condition* $\|\bar{D}^1\underset{\sim}{u}(t,\cdot)\|_\infty\leq 3U_0$ $(0\leq t\leq T)$. *Here,* $r=[s]$ *denotes the largest integer* r *with the property* $r\leq s$.

Remark. Suppose a solution $\underset{\sim}{u}\in E_T^L$ of (N.P) exist. Then, by differentiating the equation with respect to the variable t several times, we can a priori determine $\underset{\sim}{u}^p=\partial_t^p\underset{\sim}{u}(0, x)$ from the initial data $\underset{\sim}{u}^0$, $\underset{\sim}{u}^1$. Now, differentiate the boundary condition by the variable t p times and let $t=0$, then we obtain a condition for $\underset{\sim}{u}^k(x)$ $(x\in S, 0\leq k\leq p)$. For further reference, let us name this condition the p-th condition. Then the $L-2$-th order compatibility condition is the condition which requires $\underset{\sim}{u}^k(x)$ $(x\in S, 0\leq k\leq L-2)$ to satisfy the k-th condition for each k $(0\leq k\leq L-2)$.

Finally we give two examples to which we can apply our theorem.

Example 1 (cf. [8]). Put $A_i(\bar{D}_x^1 u)=\partial_i u(1+|\nabla u|^2)^{-1/2}$, $\nabla u=(\partial_1 u, \cdots, \partial_n u)$ and consider the Neumann problem (N.P):

$$(\text{N.P})\quad \begin{cases} P(u)=\partial_t^2 u-\sum_{i=1}^n \partial_i(\partial_i u(1+|\nabla u|^2)^{-1/2})+\Phi(\bar{D}^1 u)=f & \text{in } [0, T]\times\Omega, \\ Q(u)=\sum_{i=1}^n \nu_i(\partial_i u(1+|\nabla u|)^{-1/2})+\Psi(u)=g & \text{on } [0, T]\times S, \\ u(0, x)=u^0(x), \quad (\partial_t u)(0, x)=u^1(x) & \text{in } \Omega. \end{cases}$$

This is the well known classical nonlinear wave equation with Neumann $(\Psi\equiv 0)$ or the third kind boundary condition $(\Psi(0)=0$ and $\Psi\not\equiv 0)$.

Example 2. If the undeformed state Ω of a three dimensional, homogeneous, isotropic, hyperelastic material has not any stress in it, the equation of motion describing its small displacement $\underset{\sim}{u}(t, x)$ under the action of the body force $\underset{\sim}{b}(t, x, x+\nabla\underset{\sim}{u}(t, x))$ and surface force $\underset{\sim}{p}(t, x)$ of dead load type is described by the previous Neumann problem (N.P) provided that $m=n=3$, and A_{ia}, Φ_a, Ψ_a, f_a, g_a are defined as follows. Let λ, μ be the Lamé constants of this material and $\sum(\underset{\sim}{E})=(\sigma_{ij})_{1\leq i,j\leq 3}=\lambda(\text{trace }\underset{\sim}{E})\underset{\sim}{I}+2\mu\underset{\sim}{E}+o(\underset{\sim}{E})$ (as $\underset{\sim}{E}\to 0$) be its second Piola-Kirchhoff stress tensor, where $\underset{\sim}{E}$ is the strain tensor and $\underset{\sim}{I}$ is the identity matrix. Then, if $\rho(x)$ $(\geq\rho_0)$ is the density of the material,

$$A_{ia}(\nabla u)=\rho^{-1}(\sigma_{ia}+\textstyle\sum_{k=1}^3\sigma_{ka}\partial_k u_i),$$

$$\Phi_a(t, x, \underset{\sim}{u})=-(b_a(t, x, x+\underset{\sim}{u}(t, x))-b_a(t, x, x)),$$

$$\Psi_a(t, x, \underset{\sim}{u})=0,\quad f_a(t, x)=b_a(t, x, x),\quad g_a(t, x)=\rho^{-1}p_a(t, x) \quad \text{for } a=1, 2, 3.$$

Under these circumstances, the assumption (A-1) is obvious and the assumption (A-2) follows from the hyperelasticity. By mechanical experiments, we know a priori the properties:

$$\mu>0, \qquad 3\lambda+2\mu>0$$

for the small deformation of the material. Together with this properties and the famous Korn's inequality, the assumption (A-3) holds if $\|\nabla \underset{\sim}{u}\|_\infty$ is sufficiently small.

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