21. On Tight t-designs in Compact Symmetric Spaces of Rank One

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We announce the following result. For the definition of tight t-designs, see § 1.

Theorem 1. *There exists an absolute constant t_0 which satisfies the following: if X is a tight t-design in one of the complex projective spaces $P^d(C)$ $(d=4, 6, 8, \cdots)$ or the quaternion projective spaces $P^d(H)$ $(d=8, 12, 16, \cdots)$ then we have $t \leq t_0$.*

Since the corresponding results for the other compact rank 1 symmetric spaces are already obtained (see [1], [2], [5]), we have the following.

Corollary to Theorem 1. *There exists another absolute constant t_0 which satisfies the following: if X is a tight t-design in one of the connected compact rank 1 symmetric spaces of (topological) dimension $d \geq 2$, then we have $t \leq t_0$. (Here we need t_0 to be at least 11 as there exists a tight 11-design in S^{23}.)*

We expect that the actual value of t_0 in Theorem 1 can be very small (i.e., something like 5 although it may not be exactly 5). The determination of the exact value of t_0, which is very involved, will be treated in a subsequent full paper which is now being prepared by us.

§ 1. Preliminaries. Let S be a connected compact symmetric space of rank 1. That is, S is one of the following spaces: sphere S^d, projective spaces $P^d(K)$ where K is one of the real field $\boldsymbol{R}$ $(d=2, 3, 4, \cdots)$, complex field $\boldsymbol{C}$ $(d=4, 6, 8, \cdots)$, quaternion field $\boldsymbol{H}$ $(d=8, 12, 16, \cdots)$ or the Cayley octanions $\boldsymbol{O}$ $(d=16)$. Then $S=H\backslash G$ for a suitable pair of a compact Lie group G and its closed subgroup H. The space $L^2(S)$ is decomposed into the direct sum of irreducible G-spaces V_i (i.e. $L^2(S)=V_0 \oplus V_1 \oplus V_2 \oplus \cdots$) where V_i gives the i-th "spherical" representation of G. The dimension of V_i is finite and is denoted by m_i (cf. § 2).

A finite non-empty subset X of S is called a *t-design* in S if $\sum_{x \in X} f(x) = 0$ for any function $f \in V_1 \oplus V_2 \oplus \cdots \oplus V_t$. Note that for each t and each S, the existence of *t-designs* X in S is guaranteed by Seymour-Zaslavsky [17]. The reader is referred to [5], [6], [14], [16], etc. for the examples and the fundamental properties of t-designs in S.

Let $d(x, y)$ be the distance function on S, and let δ be the diameter of S, i.e., $\delta = \mathrm{Max}_{x,y \in X}\, d(x, y)$. Let $A(X) := \{d(x, y) \mid x, y \in X, x \neq y\}$. If $|A(X)|$

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$=s$, then we call X an *s-distance subset* in S. Let $e:=|A(X)\backslash\delta|$, and let $\xi=s-e$. Then ξ is either 0 or 1 according as $\delta\notin A(X)$ or $\delta\in A(X)$.

The important properties of t-designs and s-distance subsets in S are summarized by the following two propositions. (The explicit values of the m_i and the m_i^* for $P^d(K)$ are listed in § 2.)

Proposition 1.1. a) *If X is a t-design in S, then $|X|\geq 1+m_1+\cdots+m_{[t/2]}$.*

b) *If X is an s-distance subset in S, then $|X|\leq 1+m_1+\cdots+m_s$.*

c) *If X is a t-design as well as an s-distance subset in S, then $t\leq 2s$.*

d) *If X is a subset of S which satisfies both the specific condition and the equality in any one of the above three statements* a), b) *and* c), *then X also satisfies the specific condition and the equality in each of the other two of* a), b) *and* c). *If this happens, then t must be even with $t=2s=2e$. Such X is called a tight $2e$-design in S.*

Proposition 1.2. a) (Dunkl [7]). *If X is a t-design in S, then $|X|\geq 1+m_1^*+\cdots+m_{[(t-1)/2]}^*$.*

b) *If X is an s-distance subset in S with $\delta\in A(X)$, then $|X|\leq 1+m_1^*+\cdots+m_e^*$.*

c) *If X is a t-design as well as an s-distance subset in S with $\delta\in A(X)$, then $t\leq 2s-1$ $(=2e+1)$.*

d) *If X is a subset of S which satisfies both the specific condition and the equality in any one of the above three conditions* a), b) *and* c), *then X also satisfies the specific condition and the equality in each of the other two of* a), b) *and* c). *If this happens, then t must be odd with $t=2s-1=2e+1$. Such X is called a tight $(2e+1)$-design in S.*

Most of the above results are proved by linear programming methods with the help of harmonic analysis on S, by imitaing the proof for $S=S^d$ in Delsarte-Goethals-Seidel [6]. See also, Hoggar [14], Neumaier [16] (and also Bannai-Ito [3] which is now being prepared).

§ 2. Lloyd type theorem for tight *t*-designs. Tight t-designs in S^d (with $d\geq 2$) do not exist for $t\geq 6$, $\neq 7$, $\neq 11$ (see [1], [2]), and the tight 11-design is unique (with $d=23$) (see [4]). The study of tight t-designs in $P^d(\boldsymbol{R})$ is completely reduced to that of tight t-designs (with t odd) in S^d (see [5]), because, by looking at a point in $P^d(\boldsymbol{R})$ as two antipodal points in S^d, a tight $2e$-design in $P^d(\boldsymbol{R})$ corresponds to a tight $(4e+1)$-design in S^d, and a tight $(2e+1)$-design in $P^d(\boldsymbol{R})$ corresponds to a tight $(4e+3)$-design in S^d. Tight t-designs in $P^{16}(\boldsymbol{O})$ do not exist for $t\geq 6$ (see [15]). Thus, in what follows, we have only to consider $P^d(\boldsymbol{C})$ and $P^d(\boldsymbol{H})$.

Now, we list some of the important numbers and polynomials related to the harmonic analysis on $P^d(K)$. (Cf. [11], [14], [16]).

Let K be one of $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$, and let $m=(K:\boldsymbol{R})/2$. Let n be the dimension of the vector space naturally attached to the projective space. Then the topological dimension of $P^d(K)$ is given by $d=2m(n-1)$. Let $N=mn$. We refer the reader to [14, p. 240] for the definitions and the expressions of $Q_k^\xi(x)$ and $R_k^\xi(x)=Q_0^\xi(x)+Q_1^\xi(x)+\cdots+Q_k^\xi(x)$ ($\xi=0$ or 1). It is important that $R_k^\xi(x)$ (as well as $Q_k^\xi(x)$) is represented by Jacobi polynomials J_k (cf. [18]). Indeed $R_k^\xi(x)=(\text{constant}).\ J_k^{(\alpha,\beta+\xi)}(2x-1)$ with (α,β) given by $(d/2, -1/2)$ for $\boldsymbol{R}$; $(d/2, 0)$ for $\boldsymbol{C}$; and $(d/2, 1)$ for $\boldsymbol{H}$. We also note that

$$m_k=Q_k^0(1) \quad \text{and} \quad m_k^*=Q_k^1(1) \qquad \text{with}$$

$$Q_k^\xi(1)=(N)_{k+\xi-1}\cdot(N-m)_k\cdot(2k+N+\xi-1)/(m)_{k+\xi}\cdot k!,$$

where $(p)_0=1$ and $(p)_a=p(p+1)\cdots(p+a-1)$ ($a\in \boldsymbol{N}$). Note that

$$R_k^\xi(1)=(N)_{k+\xi}\cdot(N-m+1)_k/(m)_{k+\xi}\cdot k!;$$

these are the numbers

$$1+m_1+\cdots+m_k \text{ (for } \xi=0) \text{ and } 1+m_1^*+\cdots+m_k^* \text{ (for } \xi=1).$$

For later use, we also remark that

$$R_k^\xi(x)=\frac{(N)_{2k+\xi}}{(m)_{k+\xi}\cdot k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{{}_i(k+m+\xi-1)}{{}_i(2k+N+\xi-1)}x^{k-i},$$

where ${}_0(p)=1$ and ${}_a(p)=p(p-1)\cdots(p-a+1)$ ($a\in \boldsymbol{N}$).

We get the following Lloyd type theorem (cf. [1]).

Proposition 2.1. *If there exists a tight t-design in $P^d(K)$ $d\geq 2$, with $t=2e$ (resp. $t=2e+1$), then all the zeros of $R_e^0(x)$ (resp. $R_e^1(x)$) are the reciprocals of integers. Thus, if we put $f(x):=S_e^\xi(x):=x^eR_e^\xi(1/x)$, then all the zeros of $f(x)$ must be integers.*

§3. A number-theoretical result of Erdös, and its generalization. Erdös ([8], [10]) proved the following result which is a generalization of Sylvester-Schur's classical theorem:

(Erdös) If $(n+1)(n+2)\cdots(n+k)$ is a product of k consecutive integers with $n>k$ and $k\geq 3$, then it is divisible by a prime $\geq k$ with *odd* exponent.

The basic idea of his proof is briefly explained as follows. Suppose the result false. Let $n+i=a_ix_i^2$ with a_i square-free. Then, by using Sylvester-Schur, the a_i's are all distinct and all the prime factors of a_i are $<k$. Thus an upper bound of $\prod a_i$ is obtained, i.e., $\prod_{i=1}^k a_i\,|\,(k-1)!\,2^\alpha 3^\beta \prod_{5\leq p<k} p$ for suitable α and β. On the other hand, $\prod a_i$ is no less than the product of the first k square-free integers, which is $\geq(3/2)^k(k-1)!$ (if $k\geq 71$, say). This leads to a contradiction to the upper bound. (The case $k<71$ is dealt with separately.)

We get the following generalization of this result of Erdös.

Theorem 3.1. *Let α be a positive integer. Then there exists a certain function $k_0(\alpha)$ of α which satisfies the following: if $(n+2)(n+4)\cdots(n+2k)$ is a product of k consecutive odd integers with $n>2k$, then it is divisible by a prime $\geq 2k+\alpha$ with odd exponent, if $k>k_0(\alpha)$.*

Theorem 3.1 is proved by methods similar to those of Erdös, but very involved. Let $n+2_i=a_ix_i^2$ with a_i square-free. We cannot expect a very fancy upperbound of $\prod a_i$. Yet the lower bound is considerably improved, i.e., $\prod_{i=1}^k a_i\geq c^k\cdot(2k-1)!!$ for a given c if k is sufficiently (extremely) large. This is obtained by using the following lemma to show that not many a_i's are divisible by a certain prime.

Lemma 3.2. *Let a, b, N and C be any given positive integers. Then* *the number of solutions (x, y) in positive integers of the equation*

$$ax^2-by^2=\pm N \quad with \quad z<x<Cz$$

is bounded by a certain function $f(a, b, N, C)$ which is independent of z.

Lemma 3.2 is proved by using the standard results in Pell equations and related quadratic equations. (It would be very nice if we could find $k_0(\alpha)$ in Theorem 3.1 explicitly for $\alpha=1, 2, 3$, say. An explicit upperbound for $f(a, b, N, C)$ will help to find an explicit value of $k_0(\alpha)$ in Theorem 3.1.)

§4. Proof of Theorem 1 (Sketch). By Proposition 2.1, we have only to show that the polynomial $f(x)=S_e^\varepsilon(x)$ does not have all integral zeros if e is not too small. On assuming the roots *are* integers, there are two main techniques we can use to obtain a contradiction. First, the discriminant of $f(x)$ is the square of an integer. Second, all the sides of the Newton polygon corresponding to $f(x)$ (and for any prime p) must have integral slopes (cf. [2]). We divide our proof of Theorem 1 into the following four cases.

	C	H
$t=2e$	Case 3	Case 1
$t=2e+1$	Case 2	Case 4

Case 1 and Case 2. The discriminant of the classical orthogonal polynomials (including Jacobi polynomials) are calculated by Hilbert [13] (see also [18]). Using the representation of $R_e^\varepsilon(x)$ as Jacobi polynomials, we get the following diophantine equations. First, let us consider Case 1 with $e=2q$ (even). Then we get

$$(4.1) \qquad (2q+1)\underbrace{X(X+1)\cdots(X+(2q-1))}_{2q \text{ factors}}=Y^2$$

for $X=1+(d/4)$ (which is an integer) and an integer Y. For Case 1 with $e=2q+1$ (odd), we have

$$(4.2) \qquad \underbrace{(X-q)\cdots(X-1)}_{q \text{ factors}}\underbrace{(X+1)\cdots(X+q)}_{q \text{ factors}}=Y^2$$

with $X=q+1+(d/4)$ (which is an integer) and an integer Y. By slightly modifying the argument of Erdös (see [9], [10], and §3), we can easily get a contradiction if e is not too small. Case 2 is finished similarly. (We can in fact find a very small t_0 for Theorem 1.)

Case 3 and Case 4. These cases are more involved than the previous cases, as the diophantine equations which come from the discriminant of $f(x)$ are not very nice, although we expect that this condition alone should be enough to prove Theorem 1. For example, for Case 3 with $e=2q$ (even), we have

$$(4.3) \qquad \underbrace{1\cdot 2\cdot\cdots\cdot(2q)}_{2q \text{ factors}}\underbrace{(X+2)(X+4)\cdots(X+2q)}_{q \text{ factors}} \times\underbrace{(X+2q+1)(X+2q+3)\cdots(X+2q+(2q-1))}_{q \text{ factors}}=Z^2$$

with $X=(d/2)$ (which is an integer) and an integer Z. Thus, we use another technique. Again, let us consider the Case 3 with $e=2q$. Since $f(x)=$ (constant) $\sum_{i=0}^{e} a_i x^i$ with

$$a_i=\binom{e}{i}(Y-i)\cdots(Y-2q+2)(Y-2q+1)/(e-i)!$$

with $Y=4q+(d/2)$ (an integer), we get the following result by using the Newton polygon method: let p be a prime $>e$. If $p\mid(Y-i)$, then p divides $Y-i$ with exponent a multiple of $i+1$. In particular, any prime divisor $p(>e)$ of $(Y-1)(Y-3)\cdots(Y-2q+1)$ must divide the product with even exponent. This condition is actually enough to finish the proof of Theorem 1 by using Theorem 3.1 together with a slight modification of the result of Erdös mentioned at the beginning of § 3, but the value of t_0 (in Theorem 1) obtained by this proof is extremely large. The following alternative proof of Theorem 1 gives a very small t_0. Again we consider Case 3 with $e=2q$. The important fact here is that the factor $(Y-1)(Y-3)\cdots(Y-2q+1)$ is exactly the factor $(x+2q+1)\cdots(x+2q+(2q-1))$ in (4.3). This fact is very useful when applying the method of Erdös [8, 9, 10] to show that (4.3) has no integral solutions (if any prime $p>e$ which divides $(X+2q+1)\cdots(X+2q+(2q-1))$ divides it with even exponent). In fact, by putting $x+1=a_i x_i^2$ with a_i square-free in (4.3), we evaluate $\prod a_i$ from above and below, and get a contradiction if t is not too small. The other cases: Case 3 with $e=2q+1$ and Case 4 are dealt with similarly. (We can in fact find a very small t_0 for Theorem 1.)

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