87. A Note on the Spaces of Self Homotopy Equivalences for Fibre Spaces

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1. Introduction. Throughout this note, we shall work within the category of compactly generated Hausdorff spaces which will be simply called spaces. Let X and Y be spaces with base points $x_0$ and $y_0$ respectively. We denote by $\text{map}(X, Y)$ and $\text{map}_0(X, Y)$ the space of maps of X to Y and the space of maps of $(X, x_0)$ to $(Y, y_0)$ respectively. Moreover, when $k$ is a map of $X$ to $Y$, we denote by $\text{map}(X, Y; k)$ the path component of $k$ in $\text{map}(X, Y)$, and $\text{map}_0(X, Y; k)$ is defined similarly. A CW complex means a connected CW complex with non-degenerate base point. Let $X$ be a CW complex with base point $x_0$, $G(X)$ the space of self homotopy equivalences of $X$ and $G_0(X)$ the space of self homotopy equivalences of $(X, x_0)$. In previous papers [5], [6], [7] we studied $G_0(X)$ when $X = E$ is a fibre space of a fibration $F \rightarrow E \rightarrow B$. This paper is also concerned with $G_0(X)$ for a fibre space $X$.

2. Main results. We quote the following two theorems [5, 6].

Theorem A. Let $E$ and $B$ be CW complexes and $p : E \rightarrow B$ a fibration with fibre $F$. Let $n \geq 1$ be a given integer. If $F$ is $(n-1)$-connected and $\pi_i(B) = 0$ for every $i \geq n$, then we have the following fibration

$$G(E \mod F) \rightarrowtail G_0(E) \rightarrowtail G_0(B) \times G_0(F),$$

where $G(E \mod F)$ is the space of self fibre homotopy equivalences of $E$ leaving the fibre $F$ fixed.

Theorem B. Under the same hypothesis as above, the image of $p : G_0(E) \rightarrow G_0(B) \times G_0(F)$ is just the union of the path components in $G_0(B) \times G_0(F)$ each of which contains $(g, h)$ satisfying

$$[\chi_\omega(h)] \circ [k] = [k] \circ [g],$$

where $\chi_\omega(h)$ is a self map of $(B, b_0)$ and $k : (B, b_0) \rightarrow (B, b_0)$ is a classifying map in Allaud's sense for the fibration $F \rightarrow E \rightarrow B$.

Let $\varepsilon(X)$ denote the group $\pi_0(G_0(X))$ for a CW complex $X$ and let $R$ be a subgroup of $\varepsilon(B) \times \varepsilon(F)$ consisting of the elements $(g, [h])$ satisfying $[\chi_\omega(h)] \circ [k] = [k] \circ [g]$. Then our main result is the following

Theorem 1. Let $E$ and $B$ be CW complexes and $F \rightarrow E \rightarrow B = K(\pi, n)$ a fibration classified by a map $k : (B, b_0) \rightarrow (B, b_0)$ in Allaud's sense. Let $n \geq 1$ be a given integer. If $F$ is $n$-connected and $\pi_j(F) = 0$ for every $j \geq 2n$, then we have the following fibration:
map_0(B, G(F)) \to G_0(E) \to R \times G_0(F),

where $G_0$ denotes the path component in $G_0(F)$ containing the identity map $\text{id}_F$, and we have the following exact homotopy sequence of the above fibration for every $j \geq 0$

$$1 \to \pi_j(map_0(B, G(F))) \to \pi_j(G_0(E)) \to \pi_j(R \times G_0(F)) \to 1.$$ 

By using the fact that $G(F)$ has the same weak homotopy type as $F \times G_0(F)$ we can easily see the following corollary, which is a generalization of Nomura's theorem (Theorem 3.2 in [3]).

**Corollary.** Under the same hypothesis as Theorem 1, we have the following exact sequence

$$1 \to [B, F]_0 \to \pi_0(E) \to R \to 1.$$ 

**3. Sketch of proof.** We shall denote by $X \simeq Y$ that $X$ has the same weak homotopy type as $Y$. First we show the following

**Lemma 2.** It holds that

$$G(E \mod F) \simeq map_0(B, F) \simeq map_0(B, G(F)).$$

In fact, since we may regard $F$ as a loop space from our hypothesis (see Corollary 9.9 in [4]), there exists an $H$-map $\sigma : F \to G(F)$ such that $\sigma$ induces isomorphisms $\sigma_* : \pi_i(F) \to \pi_i(G(F))$ for every $i \leq n - 1$ by using Theorem 5.1 in [1]. Let $B'_\sigma$ and $B''_\sigma$ be an $(n-1)$-connective CW complex $(B_\sigma, n)$ of $B$ and an $(n-1)$-stage Postnikov complex of $B_\sigma$ respectively. Then we have a fibration $B' \to B'' \pi \to B''_\sigma$. By using Theorem 7 in [5] we have the following

$$G(E \mod F) \simeq \Omega map_0(B, B_\sigma; k) \simeq \Omega map_0(B, B'_{\sigma}; k'),$$

where $B \sigma \simeq k$. Furthermore, noting that $B'_\sigma$ has the same weak homotopy type as $BF$ and $B''_\sigma$ itself has the homotopy type of a loop space, we have

$$\Omega map_0(B, B'_\sigma; k') \simeq map_0(B, \Omega B'_\sigma) \simeq map_0(B, \Omega BF) \simeq map_0(B, G(F)).$$

Next we see easily the following

**Proposition 3.** Let $X$ be a CW complex and $Y$ a path connected $H$-space. Then there exists a cross-section $s : Y \to map(X, Y; l)$ for the following fibration:

$$map_0(X, Y; l) \to map(X, Y; l) \xrightarrow{\omega} Y,$$

where $\omega$ is the evaluation map at the base point $x_0$ of $X$.

We need the following

**Lemma 4.** $B'_\sigma$ has the same weak homotopy type as $BG_0(F)$.

In fact, there exists the map $Bi' : BG_0(F) \to BG(F) = B_\sigma$ induced by the inclusion $i' : G_0(F) \to G(F)$ (see [2]). Then the map $\pi \circ Bi' : BG_0(F) \to B''_\sigma$ induces the isomorphisms of homotopy groups.

**Proof of Theorem 1.** We have the following commutative diagram
\[ \Omega \text{map}_0 (B, B' ; k') \xrightarrow{\Omega (B j)_k} \Omega \text{map}_0 (B, B'' ; \pi \circ k) \xrightarrow{\Omega \pi} \Omega \text{map}_0 (B, B'' ; \pi \circ k) \]

Here it should be noticed that \( \Omega B'' \) has the same weak homotopy type as \( G_o(F) \) by Lemma 4.

Now, we regard the fibration on the left hand of the above diagram as the following fibration:

\[ \text{map}_0 (B, F) \xrightarrow{\omega} \text{map} (B, F) \xrightarrow{j} G(F) \xrightarrow{\Omega \pi} \Omega B'' \]

Thus, by Proposition 3 we see that

\[ (\Omega \omega)_* : \pi_*(\text{map}_0 (B, F ; l)) \rightarrow \pi_*(\text{map} (B, F ; l)) \]

is a monomorphism for every \( r \). On the other hand, we can easily see that the homomorphism

\[ (\Omega (B j)_k)_* : \pi_*(\Omega \text{map} (B, B' ; k')) \rightarrow \pi_*(\Omega \text{map} (B, B'' ; k)) \]

is a monomorphism for every \( r \). In other words, the homomorphism of \( \pi_*(\mathcal{G}(E \mod F)) \) into \( \pi_*(\mathcal{G}(E)) \) induced by the inclusion is a monomorphism for every \( r \). This implies that the following homotopy sequence of the fibration \( \rho \) is exact for every \( r \geq 0 \)

\[ 1 \rightarrow \pi_*(\mathcal{G}(E \mod F)) \rightarrow \pi_*(G_o(E)) \rightarrow \pi_*(R \times G_o(F)) \rightarrow 1. \]

Correction of the previous paper [5]. On p. 16, line 18, “a map of \( B \) to \( B'' \)” should be replaced by “a map of CW complex \( B \) to CW complex \( B'' \).

References