6. *Gauss Sums of Prehomogeneous Vector Spaces**)

By Akihiko GYOJA and Noriaki KAWANAKA
Department of Mathematics, Osaka University

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In the present article, we study a generalization of the classical Gauss sum which is associated with a prehomogeneous vector space, by using the micro-local analysis. Details which are omitted here will be published elsewhere.

1. Let V be a finite dimensional vector space over C and G a connected algebraic subgroup of $GL(V)$ which acts prehomogeneously on V, that is, there exists a proper algebraic subset S of V such that G acts homogeneously on $V-S$. We call such a pair (G, V) a prehomogeneous vector space. (See [12].) Hereafter we assume the following two conditions:

(1.1) G acts irreducibly on V.

(1.2) S is an (irreducible) hypersurface of V, that is, there exists an irreducible polynomial $f(v)$ such that $S=\{v \in V \mid f(v)=0\}$.

Such a prehomogeneous vector space is said to be *irreducible* and *regular*. Let $V^\vee$ be the dual space of V. Then $(G, V^\vee)$ is also an irreducible, regular prehomogeneous vector space. We define $S^\vee$ and $f^\vee$ in the same way as S and f. Let $\langle\, ,\rangle$ be the natural pairing of $V^\vee$ and V. Let $V^\vee \xleftarrow{\mathrm{pr}^\vee} V^\vee \times V \xrightarrow{\mathrm{pr}} V$ be the projections and $j: V-S \to V$, $j^\vee: V^\vee - S^\vee \to V^\vee$ the inclusion mappings. Let $n=\dim V$ and $d=\deg f$. It is known that there exists a polynomial $b(s)$ such that

$$f^\vee(\mathrm{grad}) f^{s+1} = b(s) f^s.$$

(See [12].) It is also known that $b(s)$ is of the form

$$b(s)=b_0 \prod_{j=1}^{d} (s+\alpha_j) \qquad (\alpha_j \in \boldsymbol{Q}, \alpha_j>0),$$

([6]). Let

$$b^{\exp}(t)=\prod_{j=1}^{d} (t-\exp(2\pi\sqrt{-1}\,\alpha_j)).$$

Then we can show that

$$b^{\exp}=\prod_{l\geq 1} \Phi_l^{m(l)} \tag{1.3}$$

with some non-negative integers $m(l)$. Here Φ_l denotes the l-th cyclotomic polynomial.

2. By a classification [12] of irreducible, regular prehomogeneous vector spaces, we see that (G, V) has a natural $\boldsymbol{Z}$-structure. If p is a sufficiently large prime number, we can get an irreducible, regular prehomogeneous vector space defined over $\boldsymbol{F}_p$, by the reduction modulo p, which

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we shall denote by the same letter (G, V). (There is no fear of confusion.)

Let $\boldsymbol{F}_q$ be a finite extension of $\boldsymbol{F}_p$, $\psi: \boldsymbol{F}_q \to \boldsymbol{C}^\times$ a non-trivial additive character and $\chi: \boldsymbol{F}_q \to \boldsymbol{C}^\times$ a multiplicative character. We define $\chi(0)=0$. We denote the order of $\chi \in \mathrm{Hom}(\boldsymbol{F}_q^\times, \boldsymbol{C}^\times)$ by $\mathrm{ord}(\chi)$. Let

(2.1) $\mathcal{F}_\psi[\chi \circ f](v^\vee) = \sum_{v \in V(\boldsymbol{F}_q)} \chi(f(v))\psi(\langle v^\vee, v\rangle) \qquad (v^\vee \in V^\vee(\boldsymbol{F}_q)).$

We call this sum the *Gauss sum* of prehomogeneous vector space (G, V). Such sums were first investigated by Z. Chen [3], and independently by the second named author [9] (in connection with the representation theory of finite reductive groups).

3. Conjecture. If p is sufficiently large,

$$\mathcal{F}_\psi[\chi \circ f](v^\vee) = \varepsilon(\chi, \psi) q^{(n-m(\mathrm{ord}\,\chi))/2} (\chi^{-1} \otimes \theta)(f^\vee(v^\vee)) \qquad (v^\vee \in (V^\vee - S^\vee)(\boldsymbol{F}_q)),$$

where $\varepsilon(\chi, \psi)$ is an algebraic number with absolute value one (with respect to any archimedean valuation) and

$$\theta = \begin{cases} \text{trivial character}, & \text{if } n/d \in \boldsymbol{Z} \\ \text{Legendre symbol}, & \text{if } n/d \in 1/2 + \boldsymbol{Z}. \end{cases}$$

(See (1.3) for $m(l)$.)

Remark. If χ is trivial, we can calculate $\mathcal{F}_\psi[\chi \circ f](v^\vee)$ $(v^\vee \in (V^\vee - S^\vee)(\boldsymbol{F}_q))$ explicitly using representation theory of finite reductive groups. Our conjecture together with this information implies:

The number of integral roots of $b(s)$ is equal to $\mathrm{rank}\, G - \mathrm{rank}\, Z_G(x)$ $(x \in V - S)$.

4. Theorem. (a) *If* $m(\mathrm{ord}\,\chi)=0$, *the above conjecture is true.*

(b) *If* (G, V) *does not belong to the castling class* (11) *in the table of* [12; pp. 144–147], *the above conjecture is true.*

Corollary. *If* $m(\mathrm{ord}\,\chi)=0$ *and* p *is sufficiently large,*

$$\mathcal{F}_\psi[\chi \circ f](v^\vee) = 0 \qquad (v^\vee \in S^\vee(\boldsymbol{F}_q)).$$

5. Outline of proof. We fix a prime number $l (\neq p)$ and an isomorphism $\bar{\boldsymbol{Q}}_l \cong \boldsymbol{C}$. Then χ (resp. ψ) can be regarded as an element of $\mathrm{Hom}(\boldsymbol{F}_q^\times, \bar{\boldsymbol{Q}}_l^\times)$ (resp. $\mathrm{Hom}(\boldsymbol{F}_q, \bar{\boldsymbol{Q}}_l^\times)$). Let $\mathcal{L}_\chi$ (resp. $\mathcal{L}_\psi$) be the Lang torsor (resp. the Artin-Schreier torsor) on $\boldsymbol{G}_m$ (resp. $\boldsymbol{G}_a$) associated with χ (resp. ψ). Note that $\mathcal{L}_\chi$ is a Kummer torsor. Define the (sheaf theoretical) Fourier transformation [8]

$$\mathcal{F}_\psi : D_c^b(V, \bar{\boldsymbol{Q}}_l) \longrightarrow D_c^b(V, \bar{\boldsymbol{Q}}_l)$$

by

$$\mathcal{F}_\psi[-] = \boldsymbol{R}\,\mathrm{pr}_{!}^\vee (\mathrm{pr}^*(-) \overset{L}{\otimes} \langle\ \rangle^* \mathcal{L}_\psi).$$

(See [4; (1.1.3)] for $D_c^b(-, \bar{\boldsymbol{Q}}_l)$.) The above theorem is a consequence of the following theorem.

Theorem*. *Assume that* p *is sufficiently large.*

(a*) *If* $m(\mathrm{ord}\,\chi)=0$, *then* $j_! f^* \mathcal{L}_\chi$ *is pure of weight zero.*

(b*) *Assume that* (G, V) *does not belong to the castling class* (11). *Then*

(b*1) $\mathcal{F}_\psi[j_! f^* \mathcal{L}_\chi]|_{V^\vee - S^\vee} = f^{\vee *} \mathcal{L}_{\chi^{-1} \otimes \theta}[-n]|_{V^\vee - S^\vee}$.

(b*2) $\mathcal{F}_\psi[j_! f^* \mathcal{L}_\chi]|_{V^\vee - S^\vee}$ *is pure of weight* $-m(\mathrm{ord}\,\chi)$.

The part (a*) is rather easy.

By considering the Radon transformation [1], [8] or an analogue of the Jacobi sum, the proof of the equality (b*1) can be reduced to that of an analogous equality over $(V-S)(C)$. (See [1] [5].) Then, by the Riemann-Hilbert correspondence [7], our problem can be translated to showing an equality of $\mathcal{D}$-modules, which we can do. (Here $\mathcal{D}$ is the sheaf of differential operators.)

6. Outline of the proof of (b*2). Since $j_! f^* \mathcal{L}_\chi$ is a perverse sheaf, there is a weight filtration

$$j_! f^* \mathcal{L}_\chi = \mathcal{W}_0 \supset \mathcal{W}_{-1} \supset \mathcal{W}_{-2} \supset \cdots . \tag{6.1}$$

(See [2].) Let $s=1/\mathrm{ord}\,\chi$, $\mathcal{M}'=\mathcal{D}f^s$ and $\mathcal{M}=\mathcal{M}'[1/f]$. Then from the filtration (6.1) and by the same argument as in the proof of (b*1), we can construct a filtration

$$\mathcal{M}^* = \mathcal{M}_0 \supset \mathcal{M}_{-1} \supset \mathcal{M}_{-2} \supset \cdots . \tag{6.2}$$

Here $\mathcal{M}^*$ is the dual (regular, holonomic) $\mathcal{D}$-module of $\mathcal{M}$. By using results of [2], [11] and calculations of holonomy diagrams of individual prehomogeneous vector spaces (see [10] and its references), we can prove the following lemma (except for the case of type (11)).

Lemma. (a) $\mathcal{M}_0 \supsetneqq \mathcal{M}_{-1} \supsetneqq \cdots \supsetneqq \mathcal{M}_{-m(\mathrm{ord}\,\chi)} \supsetneqq \mathcal{M}_{-m(\mathrm{ord}\,\chi)-1} = 0$.

(b) $\mathrm{Supp}\,\mathcal{E} \otimes \mathcal{M}_{-m(\mathrm{ord}\,\chi)} \supset V^\vee \times \{0\}$, *where $\mathcal{E}$ is the sheaf of micro-differential operators.*

Once this lemma is settled, the remaining is rather easy.

7. Remark. By a more detailed argument, we can determine the set of prime numbers p which should be excluded in our theorem.

8. Remark. In [13], M. Sato and T. Shintani introduced zeta functions associated with prehomogeneous vector spaces. It is natural to expect that our Gauss sums enter in functional equations of "L-functions" of prehomogeneous vector spaces. We learned from F. Sato that this is indeed the case at least when $m(\mathrm{ord}\,\chi)=0$.

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