25. A Study of a Certain Non-Conventional Operator of Principal Type

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1984)

Introduction. The purpose of this note is to report some of the properties of the first order differential operator:

(1) $$B^I = D_t + i(t^2/2 + x)D_y,$$

$D_t = -i\partial/\partial t$, $D_y = -i\partial/\partial y$, in a neighborhood of the origin in $\boldsymbol{R}^3$. Its principal symbol is given by $b^I = \tau + i(t^2/2 + x)\eta$, if (τ, ξ, η) denotes the dual variables of (t, x, y). Observe then $\{b^I, \bar{b}^I\} = -2it\eta$. Let $S^{\pm} = \{(t, x, y, \tau, \xi, \eta)\,;\, \tau = 0,\ t^2/2 + x = 0,\ \pm t\eta < 0\}$ and $S_1 = \{(t, x, y, \tau, \xi, \eta)\,;\, \tau = \eta = 0,\ \xi \neq 0\}$. The characteristic set S of B^I is connected and consists of two cones S_1 and $S_2 = S^+ \cup S^- \cup S^0$, where $S^0 = \{(0, 0, y, 0, \xi, \eta)\,;\, \eta \neq 0\}$. A noteworthy fact is that $\{b^I, \bar{b}^I\}/2i$ changes sign on S_2 near S^0. In this sense, the operator B^I does not microlocally fall in the class of operators conventionally studied ([2], [3], [6]). However, we can show the following

Theorem 1. *Let*

(2) $$B^I u = f, \quad u \in \mathcal{D}'(\boldsymbol{R}^3), \quad f \in \mathcal{E}'(\boldsymbol{R}^3),$$

with supp f *in a small neighborhood of the origin. If* $(t_0, x_0, y_0, \tau_0, \xi_0, \eta_0) \in WF(u) \setminus WF(f)$, $\eta_0 \neq 0$, *is in a conic neighborhood* Γ *of* $(0, 0, 0, 0, 0, \eta_0/|\eta_0|)$, *then* $\tau_0 = 0$ *and* $(t_0, x_0, y_0, 0, \xi_0, \eta_0) \in S^+ \cup S^0$.

Note that the general theory [4] assures $WF(u) \setminus WF(f) \subset S$ so that $\tau_0 = 0$ is immediate. A proof of Theorem 1 will be given in §1. We will considerably make use of the particular form of the operator B^I. In this respect, we also include here a result on the equation $B^I u = 0$. Let $u \in \mathcal{D}'(\boldsymbol{R}^3)$. Introduce the quantities:

$$t^*(x, y\,;\, u) = \sup\{t\,;\, (t, x, y) \in \operatorname{supp} u\}, \qquad (x, y) \in \boldsymbol{R}^2,$$
$$y^*(x, t\,;\, u) = \sup\{y\,;\, (t, x, y) \in \operatorname{supp} u\}, \qquad (x, t) \in \boldsymbol{R}^2,$$

adopting the convention $\sup \phi = -\infty$. Replacing sup by inf, we define $t_*(x, y\,;\, u)$ and $y_*(x, t\,;\, u)$ with $\inf \phi = +\infty$. Note $t^*(x, y\,;\, u)$ and $y^*(x, t\,;\, u)$ (resp. $t_*(x, y\,;\, u)$ and $y_*(x, t\,;\, u)$) are upper (resp. lower) semicontinuous.

Theorem 2. *Let* $u \in \mathcal{D}'(\boldsymbol{R}^3)$ *satisfy* $B^I u = 0$. *Assume one of the quantities* $t^*(x, y\,;\, u)$, $y^*(x, t\,;\, u)$, $-t_*(x, y\,;\, u)$ *and* $-y_*(x, t\,;\, u)$ *take a finite local maximum. Then* u *vanishes identically.*

A proof will be given in §2.

Before ending Introduction, we briefly indicate our motivation in order to arouse (hopefully) a general interest. According to [1], [5], [8], if a vector field X in $\boldsymbol{R}^{n+1}$ and a hypersurface H are in general position, then we can introduce a local coordinate system $x_1, \cdots, x_n, x_{n+1}$, such that H is represented by $x_{n+1}=0$ and X takes the form of one of the following X_l, $l=1, \cdots, n+1$: $X_l=x_n\partial_{n+1}+\cdots+x_l\partial_{l+1}+\partial_l$, $l=1, \cdots, n$, and $X_{n+1}=\partial_{n+1}$, where $\partial_j=\partial/\partial x_j$, $j=1, \cdots, n+1$. This suggests that the pseudo-differential operator B with the principal symbol $-x_n|\xi|+i\ (\sum_{j=l+1}^n x_{j-1}\xi_j+\xi_l)$ $(l\leq n)$ is a typical local model of the Calderon operator [7] associated to a second order elliptic oblique derivative problem. Except [5], only the cases $l\geq n$ have thoroughly been studied and no pseudo-differential approaches for the cases $l\leq n-1$ seem to have ever been made [6], [7]. The operator (1) in this note is a still simplified microlocal model for $l=n-1$.

1. *Proof of Theorem* 1. First we introduce an auxiliary function $\beta(t, r, x)=\int_r^t (s^2/2+x)ds$. When $x\geq 0$, $\beta(t, r, x)\leq 0$ if and only if $t\leq r$. Now let $x<0$. Denote by $I_0^-(t)$ the component, which contains t, of the set of r for which $\beta(t, r, x)\leq 0$. $I_0^-(t)$ reduces to a point when and only when $t=-\sqrt{-2x}$ and $I_0^-(-\sqrt{-2x})=\{-\sqrt{-2x}\}$. $\operatorname{Min} I_0^-(t)=t$ when $|t|>\sqrt{-2x}$ and $\operatorname{Max} I_0^-(t)=t$ when $|t|<\sqrt{-2x}$. If $t=\sqrt{-2x}$, then t is an interior point of $I_0^-(t)$. In (2), we may assume f and u are both supported in a small neighborhood of (t_0, x_0, y_0). We may further suppose that both $WF(u)$ and $WF(f)$ are contained in Γ and that there is a conic neighborhood Γ_1 of $(t_0, x_0, y_0, 0, \xi_0, \eta_0)$ such that $\Gamma_1\cap WF(f)=\phi$ and $\Gamma_1\subset\Gamma$. Fourier transforming with respect to y in (2), we get

$$\begin{aligned}(3)\qquad &\hat{u}(t, x, \eta)-e^{\beta(t,r,x)\eta}\hat{u}(r, x, \eta)\\ &\quad=i\int_r^t\iint e^{i(s\sigma+x\xi)+\beta(t,s,x)\eta}F(\sigma, \xi, \eta)d\sigma d\xi ds,\end{aligned}$$

where F is the Fourier transform of f. Assume $x_0<0$. Let $\eta_0>0$. If $t_0\neq-\sqrt{-2x_0}$, we can take $r_0\in I_0^-(t_0)\backslash\{t_0\}$ such that for $r=r_0+\rho$, $t=t_0+\rho$, $|\rho|<\varepsilon_1$ small enough, $r\in I_0^-(t)\backslash\{t\}$. We may then assume $u(r, x, y)=0$ for $r\leq r_0+\varepsilon_1$ so that the second term on the left hand side of (3) vanishes. Let Σ and Σ_1 be the projection to the fiber variables of Γ and Γ_1, respectively. Note $(\sigma, \xi, \eta)\in\Sigma$ if and only if $|\sigma|+|\xi|<a\eta$ for some $a>0$. Divide the inner integral on the right hand side of (3) into one over Σ_1 and another over $\Sigma\backslash\Sigma_1$. Since $WF(f)\cap\Gamma_1=\phi$ and $(0, \xi_0, \eta_0)\in\Sigma_1$, we see $(t_0, x_0, y_0, 0, \xi_0, \eta_0)\notin WF(u)$. Similarly, unless $t_0=\sqrt{-2x_0}$, $(t_0, x_0, y_0, 0, \xi_0, \eta_0)\notin WF(f)$, $\eta_0<0$, implies $(t_0, x_0, y_0, 0, \xi_0, \eta_0)\notin WF(u)$. On the other hand, when $x>0$ or when $x=0$ and $t\neq 0$, it is clear that $(t, x, y, 0, \xi, \eta)\notin WF(f)$ always implies $(t, x, y, 0, \xi, \eta)\notin WF(u)$ when $\eta\neq 0$. This completes the proof of Theorem 1.

2. *Proof of Theorem* 2. Theorem 2 is an easy consequence of the following

Lemma. *Let* $u \in \mathcal{D}'(\boldsymbol{R}^3)$ *satisfy the equation* $B^I u = 0$. *Then* $N(\operatorname{supp} u) \subset S_1$. *Furthermore, if* (t_0, x_0, y_0) *is a boundary point of* $\operatorname{supp} u$, *then there is a neighborhood* U *of this point such that each component of* $U \setminus \operatorname{supp} u$ *lies in the half spaces* $x \geq x_0$ *or* $x \leq x_0$.

In the above, $N(\operatorname{supp} u)$ is the whole normal set of $\operatorname{supp} u$, i.e., the union of the normal set $N_e(\operatorname{supp} u)$ and the interior normal set $N_i(\operatorname{supp} u)$ ([4] pp. 300–301). To prove Lemma, let $\mathcal{C}$ be, as in [4], the smallest subset of $C^\infty(T^*(\boldsymbol{R}^3)\setminus 0)$ which contains all the C^∞ functions vanishing on S and is closed under the Poisson bracket. Then $\mathcal{C} = \{\tau, (t^2/2 + x)\eta, t\eta, \eta\}$. Therefore, Theorem 8.6.6 in [4] implies the first assertion. The second assertion is immediate from Proposition 8.5.8 in [4].

References

[1] V. I. Arnol'd: Local problems of analysis. Moscow Univ. Math. Bull., **25**, 77–80 (1970).

[2] J. Duistermaat and J. Sjostrand: A global construction for pseudo-differential operators with non-involutive characteristics. Invent. math., **20**, 209–225 (1973).

[3] L. Hörmander: Subelliptic operators. Sem. on Singularities of Solutions of Linear PDE. ed. by L. Hörmander, Ann. Math. Studies, no. 91, Princeton University Press, pp. 127–208 (1979).

[4] ——: The Analysis of Linear Partial Differential Operators I. Springer-V. (1983).

[5] V. G. Maz'ja: On a degenerating problem with directional derivative. Math. USSR Sbornik, **16**, 429–469 (1972).

[6] A. Melin and J. Sjostrand: Fourier integral operators with complex phase functions and parametrix for an interior boundary value problem. Comm. in PDE, **1**, 313–400 (1976).

[7] F. Treves: Introduction to Pseudodifferential and Fourier Operators. vol. 1, Plenum (1980).

[8] S. M. Vishik: Vector fields in the neighborhood of the boundary of a manifold. Moscow Univ. Math. Bull., **27**, 13–19 (1972).