24. *Path Integral for Some Systems of Partial Differential Equations*

By Hiroshi WATANABE
Department of Mathematics, Tokyo Metropolitan University

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§ 1. **Introduction.** The Feynman path integral has been discussed by many authors and has various mathematical formulations (see [1]–[3] and the references cited in [2]). In each case we find a new Feynman-Kac formula extended to the quantum mechanical wave equations by generalizing the notion of measure. Albeverio and Höegh-Krohn [1] considered an analytic continuation of the characteristic function of the Wiener measure to integrate formally some functionals and gave fundamental solutions for Schrödinger equations in the form of the path integral.

In this note we shall propose a generalization of [1] to a wider class of operators which involves some hyperbolic systems and Schrödinger operators as special cases.

§ 2. **Formulation of the path integral.** We write $\partial=(\partial/\partial x_1, \cdots, \partial/\partial x_d)$, $D=-i\partial$, $D_\tau=-i\partial/\partial\tau$ and $\langle p\rangle=(1+|p|^2)^{1/2}$. We consider a free Hamiltonian $H(D)$ having the following properties: 1) The symbol $H(p)$ of the free Hamiltonian is an $r\times r$ matrix-valued continuous function on $\boldsymbol{R}^d$ satisfying

(2.1) $$|H(p)|<C\langle p\rangle^m, \qquad p\in \boldsymbol{R}^d,$$

for some constants $C>0$ and $m\geq 1$; and 2) for any $p\in\boldsymbol{R}^d$, $H(p)$ is a dissipative matrix, that is,

(2.2) $$\operatorname{Im}(H(p)X, X)>0, \qquad X\in \boldsymbol{C}^r,$$

where $(\cdot,\cdot)$ is the inner product in $\boldsymbol{C}^r$.

The Dirac operator $\sum_{j=1}^3 \alpha_j D_j+\mu\beta$ and the operators $-\Delta/2$ and $-i\Delta/2$, for example, have these properties with $m=1, 2$ and 2, respectively.

In the following we regard the Lebesgue space $L^m=L^m([s,t];\boldsymbol{R}^d)$ as a measurable space whose σ-algebra is the set of all Borel sets in L^m. Put $m'=m/(m-1)$. We say that a map $\Psi(\cdot): L^{m'}([s,t];\boldsymbol{R}^d)\to\boldsymbol{C}^r$ belongs to $F^m(s,t)$ if Ψ is a characteristic function of some $\boldsymbol{C}^r$-valued bounded measure $\hat{\Psi}$ on $L^m([s,t];\boldsymbol{R}^d)$, that is,

$$\Psi(v)=\int_{L^m} e^{-i\langle v,\xi\rangle}\hat{\Psi}(d\xi), \qquad v\in L^{m'}([s,t];\boldsymbol{R}^d),$$

where $\langle v,\xi\rangle$ denotes the pairing of $v\in L^{m'}$ and $\xi\in L^m$.

Let us introduce the symbol $\prod_s^\tau \exp[iH(\xi(\sigma))d\sigma]$, $\xi\in L^m([s,t];\boldsymbol{R}^d)$, which denotes the solution $K(s,\tau)$ of the Cauchy problem

$$D_\tau K(s,\tau)=H(\xi(\tau))K(s,\tau),\quad \tau\in[s,t],\quad K(s,s)=I,$$

where I is the unit matrix of size r. The existence of $K(s,\tau)$ follows from (2.1) and (2.2).

Definition 2.1. Let $H(D)$ be a free Hamiltonian satisfying (2.1) and (2.2). We define the path integral $\int \Psi(v)\mu_H(s,t\,;dv)$ of $\Psi(\cdot)\in F^m(s,t)$ with respect to the *generalized measure* $\mu_H(s,t\,;dv)$ by the following equality: $\int \Psi(v)\mu_H(s,t\,;dv)=\int \prod_s^t \exp[iH(\xi(\sigma))d\sigma]\hat{\Psi}(d\xi)$.

Note that $\left|\int \Psi(v)\mu_H(s,t\,;dv)\right|\le|\hat{\Psi}|$, where $|\hat{\Psi}|$ denotes the total variation of $\hat{\Psi}$. In the case that $H=-\Delta/2$, our definition agrees with that in [1].

§3. Scalar potentials. Let $H(D)$ be a free Hamiltonian whose symbol is a polynomial of order m and satisfies (2.2). We consider the following Cauchy problem

(3.1) $\quad D_\tau u(\tau,x)=(H(D)+A(\tau,x)I)u(\tau,x) \quad$ on $[s,t]\times \boldsymbol{R}^d$,

(3.1)′ $\quad u(s,x)=u_0(x) \quad$ on $\boldsymbol{R}^d$.

We assume that the scalar potential $A(\tau,x)$ is written as

$$A(\tau,x)=\int e^{iq\tau+ipx}\hat{A}(dqdp),\quad \tau\in[s,t],\quad x\in\boldsymbol{R}^d, \tag{3.2}$$

for some $\boldsymbol{C}$-valued bounded measure $\hat{A}(dqdp)$ on $\boldsymbol{R}\times\boldsymbol{R}^d$ with

$$\int|\hat{A}(dqdp)|\langle p\rangle^m<\infty. \tag{3.3}$$

We shall write the solution of (3.1) and (3.1)′ in the form of the path integral defined in §2. For $\tau\in[s,t]$, $x\in\boldsymbol{R}^d$ and $v\in L^{m'}([s,\tau]\,;\boldsymbol{R}^d)$, $m'=m/(m-1)$, we put

$$\Phi(\tau,x\,;v)=\exp\left[i\int_s^\tau A(\sigma,x(\sigma))d\sigma\right]u_0(x(s)), \tag{3.4}$$

where $x(\sigma)=x-\int_\sigma^\tau v(\theta)d\theta$. Then we have

Theorem 3.1. *If the initial value $u_0(x)$ is written as*

$$u_0(x)=\int e^{ipx}\hat{u}_0(dp),\qquad x\in\boldsymbol{R}^d, \tag{3.5}$$

for some $\boldsymbol{C}^r$-valued bounded measure $\hat{u}_0$ on $\boldsymbol{R}^d$ satisfying

$$\int|\hat{u}_0(dp)|\langle p\rangle^m<\infty, \tag{3.6}$$

then, for each $\tau\in[s,t]$ and each $x\in\boldsymbol{R}^d$, $\Phi(\tau,x\,;\cdot)$ belongs to $F^m(s,\tau)$, and $u(\tau,x)$ defined by $u(\tau,x)=\int\Phi(\tau,x\,;v)\mu_H(s,\tau\,;dv)$, $\tau\in[s,t]$, $x\in\boldsymbol{R}^d$, satisfies (3.1) *and* (3.1)′.

Let us interpret the convergence of the product integral considered in [4] from our view point. To this end we further assume that

(3.2)′ $\quad A(\tau,x)=A(x)=\int e^{ipx}\hat{A}(dp),\quad \tau\in[s,t],\quad x\in\boldsymbol{R}^d,$

for a $\boldsymbol{C}$-valued bounded measure $\hat{A}(dp)$ on $\boldsymbol{R}^d$ which is independent of τ and satisfies

$$(3.3)' \qquad \int \langle p\rangle^k |\hat{A}(dp)| < \infty, \qquad k \geq 0.$$

For each $\tau \in \boldsymbol{R}$ we define the integral operator $\Gamma(\tau)$ on $\mathcal{S}(\boldsymbol{R}^d)$ by

$$\Gamma(\tau)\varphi(x) = \frac{1}{2\pi i}\int dp \int dy \exp[i(x-y)p + i\tau(H(p)+A(x)I)]\varphi(y), \quad \varphi \in \mathcal{S}(\boldsymbol{R}^d).$$

Note that $\Gamma(\tau)$ maps $\mathcal{S}(\boldsymbol{R}^d)$ into $\mathcal{S}(\boldsymbol{R}^d)$. Define the cylinder approximation Φ_Δ of Φ by

$$(3.7) \qquad \Phi_\Delta(\tau, x; v) = \exp[i\textstyle\sum_{j=0}^{n-1} A(x(\tau_j))(\tau_{j+1}-\tau_j)]u_0(x(s))$$

for $\tau \in [s,t]$, $x \in \boldsymbol{R}^d$, $v \in L^{m'}([s,t]; \boldsymbol{R}^d)$ and for a subdivision Δ, $s=\tau_0<\tau_1<\cdots<\tau_n=\tau$, of the interval $[s,\tau]$, where $x(\sigma) = x - \int_\sigma^\tau v(\theta)d\theta$.

Proposition 3.2. *Let $H(p)$ be a polynomial of order m satisfying* (2.2) *and assume* (3.2)′ *and* (3.3)′. *If $u_0 \in \mathcal{S}(\boldsymbol{R}^d)$, we have, for $\tau \in [s,t]$, $x \in \boldsymbol{R}^d$ and for a subdivision Δ, $s=\tau_0<\tau_1<\cdots<\tau_n=\tau$, of the interval $[s,\tau]$,*

$$(3.8) \qquad \int \Phi_\Delta(\tau, x; v)\mu_H(s,\tau; dv) = \Gamma(\tau-\tau_{n-1})\cdots\Gamma(\tau_1-s)u_0(x) \quad \textit{and}$$

$$(3.9) \qquad \int \Phi(\tau, x; v)\mu_H(s,\tau; dv) = \lim_{|\Delta|\to 0}\int \Phi_\Delta(\tau, x; v)\mu_H(s,\tau; dv),$$

where Φ and Φ_Δ are defined by (3.4) *and* (3.7), *respectively, and* $|\Delta| = \max_{0\le j\le n-1}(\tau_{j+1}-\tau_j)$.

The equality (3.9) follows from the fact that $\lim_{|\Delta|\to 0}\int f(\xi)\hat{\Phi}_\Delta(\tau, x; d\xi) = \int f(\xi)\hat{\Phi}(\tau, x; d\xi)$, for an arbitrary $r\times r$ matrix-valued strongly continuous function $f(\xi)$ on $L^m([s,\tau]; \boldsymbol{R}^d)$.

§4. **Vector potentials.** Throughout this section we assume that $H(p) = \sum_{j=1}^d \omega_j p_j + \omega_0$ for $r\times r$ constant matrices ω_j $(j=0,1,\cdots,d)$ and that $H(p)$ satisfies (2.2). A typical example is the Dirac operator.

We consider the following Cauchy problem

$$(4.1) \qquad D_\tau u(\tau,x) = \{H(D-A(\tau,x)) + A_0(\tau,x)I\}u(\tau,x) \qquad \text{on } [s,t]\times\boldsymbol{R}^d,$$

$$(4.1)' \qquad u(s,x) = u_0(x) \qquad \text{on } \boldsymbol{R}^d,$$

where A and A_0 are functions defined on $[s,t]\times\boldsymbol{R}^d$ with values in $\boldsymbol{C}^d$ and $\boldsymbol{C}^1$, respectively. We assume that A satisfies (3.2) and (3.3) with $m=1$, and that A_0 satisfies (3.2) and (3.3) with A and m replaced by A_0 and 1, respectively. Suppose that u_0 is of the form (3.5) and satisfies (3.6) with $m=1$. In order to give the solution of (4.1) and (4.1)′ in the form of the path integral, we define

$$(4.2) \qquad \tilde{\Phi}(\tau, x; v) = \exp\left[i\int_s^\tau A(\sigma, x(\sigma))v(\sigma)d\sigma + i\int_s^\tau A_0(\sigma, x(\sigma))d\sigma\right]u_0(x(s))$$

for $\tau \in [s,t]$, $x \in \boldsymbol{R}^d$ and $v \in L^\infty([s,\tau]; \boldsymbol{R}^d)$, where $x(\sigma) = x - \int_\sigma^\tau v(\theta)d\theta$

Since $\tilde{\Phi}$ does not belong to $F^1(s,\tau)$, in order to integrate $\tilde{\Phi}$ with respect to $\mu_H(s,\tau;dv)$ we expand the right hand side of (4.2) as

$$\tilde{\Phi}(\tau,x;v)=\sum_{N=0}^{\infty}\frac{1}{N!}\int_s^\tau id\sigma_1\cdots\int_s^\tau id\sigma_N\sum_{j_1=1}^{d}\cdots\sum_{j_N=1}^{d}$$

$$\prod_{k=1}^{N}A_{j_k}(\sigma_k,x(\sigma_k))v_{j_k}(\sigma_k)\cdot\exp\left[i\int_s^\tau A_0(\sigma,x(\sigma))d\sigma\right]u_0(x(s)).$$

For fixed $\sigma_1,\cdots,\sigma_N\in[s,\tau]$ and $j_1,\cdots,j_N\in\{1,2,\cdots,d\}$, we put

$$\tilde{\Phi}_0(\tau,x;v)=\prod_{k=1}^{N}A_{j_k}(\sigma_k,x(\sigma_k))\cdot\exp\left[i\int_s^\tau A_0(\sigma,x(\sigma))d\sigma\right]u_0(x(s)).$$

We can easily show that $\tilde{\Phi}_0$ belongs to $F^1(s,\tau)$. Moreover, as a result of the following proposition, the path integral

$$\int\prod_{k=1}^{N}v_{j_k}(\sigma_k)\cdot\tilde{\Phi}_0(\tau,x;v)\mu_H(s,\tau;dv)$$

$$=\lim_{\varepsilon\to0}\int\prod_{k=1}^{N}\frac{1}{\varepsilon}(x_{j_k}(\sigma_k+\varepsilon)-x_{j_k}(\sigma_k))\cdot\tilde{\Phi}_0(\tau,x;v)\mu_H(s,\tau;dv)$$

exists, though $v(\sigma)$ has no definite value at one point $\sigma\in[s,\tau]$ when $v\in L^\infty([s,\tau];\boldsymbol{R}^d)$ is fixed.

Proposition 4.1. *Let $\Psi(\cdot)\in F^1(s,\tau)$. For $s\leq\sigma_1\leq\cdots\leq\sigma_N\leq\tau$ and $j_1,\cdots,j_N\in\{1,\cdots,d\}$, we have*

$$\lim_{\varepsilon\to0}\int\prod_{k=1}^{N}\frac{1}{\varepsilon}(x_{j_k}(\sigma_k+\varepsilon)-x_{j_k}(\sigma_k))\cdot\Psi(v)\mu_H(s,\tau;dv)$$

$$=\int\prod\nolimits_{\sigma_N}^{\tau}\exp[iH(\xi(\theta))d\theta]\cdot\omega_{j_N}\prod\nolimits_{\sigma_{N-1}}^{\sigma_N}\exp[iH(\xi(\theta))d\theta]\cdot\omega_{j_{N-1}}$$

$$\cdots\omega_{j_1}\prod\nolimits_{s}^{\sigma_1}\exp[iH(\xi(\theta))d\theta]\cdot\hat{\Psi}(d\xi),$$

where $x(\sigma)=x-\int_\sigma^\tau v(\theta)d\theta$.

Note that

$$\left|\int\prod_{k=1}^{N}v_{j_k}(\sigma_k)\cdot\Psi(v)\mu_H(s,\tau;dv)\right|\leq c^N|\hat{\Psi}|,\tag{4.3}$$

where $c\geq|\omega_j|$ for $j=1,\cdots,d$. By virtue of (4.3) we can show that $\tilde{\Phi}(\tau,x;v)$ defined by (4.2) is integrable with respect to $\mu_H(s,\tau;dv)$. Moreover we have

Theorem 4.2. *The function $u(\tau,x)$ defined by*

$$u(\tau,x)=\int\tilde{\Phi}(\tau,x;v)\mu_H(s,\tau;dv),\quad\tau\in[s,t],\quad x\in\boldsymbol{R}^d,$$

satisfies (4.1) *and* (4.1)′.

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