22. *On the Telegraph Equation and the Toda Equation*

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§1. **Summary.** We can solve the Toda equation with two time variables

(1.1) $$XY \log t_n = t_{n+1}t_{n-1}/t_n^2$$

($X=\partial/\partial x$, $Y=\partial/\partial y$, $t_n=t_n(x,y)$) using solutions of the telegraph equation

(1.2) $$(XY+1)u_n=0.$$

Rational solutions, Bessel function solutions and solutions which are expressed by hypergeometric functions with two variables are obtained.

§2. **Bäcklund transformation.** When t_n satisfies (1.1)

(2.1) $$r_n = XY \log t_n, \qquad s_n = Y \log t_{n-1}/t_n$$

satisfies

(2.2) $$Yr_n = r_n(s_n - s_{n+1}), \qquad Xs_n = r_{n-1} - r_n.$$

Let us introduce the following triple of partial differential operators

(2.3) $$M_n = XY + s_{n+1}X + r_n, \quad X_n = -r_n^{-1}X, \quad Y_n = Y + s_{n+1}.$$

Define

(2.4) $$T=\{u_n\,;\, M_0u_0=0,\ u_{n+1}=Y_nu_n\ (n \geqslant 0),\ u_{n-1}=X_nu_n\ (n \leqslant 0)\}.$$

We can show

Theorem 2.1 (Bäcklund transformation). *If $u_n \in T$ then we have $M_nu_n=0$, $u_{n+1}=Y_nu_n$, $u_{n-1}=X_nu_n$ $(n=0, \pm 1, \pm 2, \cdots)$ and $\tau_n=u_nt_n$ satisfies the Toda equation* (1.1).

We can obtain all solutions of the Toda equation (2.2) with separated form $r_n=f(n)g(x,y)$. $f(n)$ must be a polynomial in n of order 2 and our solutions are

(i) $r_n=(n-\alpha)(n-\beta)a'(x)b'(y)(a(x)+b(y))^{-2}$,

(ii) $r_n=(n-\alpha)a(x)b(y)$, (iii) $r_n=a(x)b(y)$,

where α and β are arbitrary constants and $a(x)$ and $b(y)$ are arbitrary functions. In this note we only treat the Bäcklund transforms of the simplest solutions (iii).

§3. **One-parameter groups on *T*.** No loss of generality we can assume that $a(x)=b(y)=1$. In this case we have

(3.1) $$t_n=e^{xy}, \quad r_n=1, \quad s_n=0,$$

(3.2) $$M_n=M=XY+1, \quad X_n=-X, \quad Y_n=Y.$$

We can determine all of the first order partial differential operators $D=a(x,y)X+b(x,y)Y+c(x,y)$ which commute with M (modulo M).

Theorem 3.1. *Dimension of the vector space* $\{D=aX+bY+c;\ MD-DM=(a_x+b_y)M\}$ *is* 4. *The bases are* $X, Y, Z=yY-xX, 1$. *If* $u\in\ker M$ *then* $Xu, Yu, Zu\in\ker M$.

We can construct three one-parameter groups of linear transformations and a finite group which keep T invariant.

Theorem 3.2 (Main theorem). *If* $u_n\in T$ *then*

$$\tilde{X}(\lambda)u_n(x,y)=u_n(x+\lambda,y),\qquad \tilde{Y}(\mu)u_n(x,y)=u_n(x,y+\mu),\tag{3.3}$$
$$\tilde{Z}_n(\nu)u_n(x,y)=e^{n\nu}u_n(e^{-\nu}x, e^{\nu}y),$$
$$Ru_n(x,y)=(-1)^n u_{-n}(y,x)\tag{3.4}$$

belong to T. $\tilde{X}(\lambda)$, $\tilde{Y}(\mu)$ *and* $\tilde{Z}_n(\nu)$ *are one-parameter groups of linear transformations with generators* X, Y *and* $Z_n=yY-xX+n$ *respectively. Each of these one-parameter groups and corresponding generators keep* $\ker M$ *invariant.* $\{R^2=id., R\}$ *is a finite group.*

We can show the following commutation relations.

Theorem 3.3 (Commutation relations). *For any values of complex numbers* λ, μ *and* ν *we have*

$$\tilde{X}(\lambda)\tilde{Y}(\mu)=\tilde{Y}(\mu)\tilde{X}(\lambda),\qquad \tilde{X}(\lambda)\tilde{Z}_n(\nu)=\tilde{Z}_n(\nu)\tilde{X}(e^{-\nu}\lambda),\tag{3.5}$$
$$\tilde{Y}(\mu)\tilde{Z}_n(\nu)=\tilde{Z}_n(\nu)\tilde{Y}(e^{\nu}\mu),$$
$$\tilde{X}(\lambda)Y=Y\tilde{X}(\lambda),\qquad \tilde{X}(\lambda)Z_n=(Z_n-\lambda X)\tilde{X}(\lambda),\tag{3.6}$$
$$\tilde{Y}(\mu)Z_n=(Z_n+\mu Y)\tilde{Y}(\mu),\qquad \tilde{Y}(\mu)X=X\tilde{Y}(\mu),$$
$$\tilde{Z}_n(\nu)X=e^{\nu}X\tilde{Z}_n(\nu),\qquad \tilde{Z}_n(\nu)Y=e^{-\nu}Y\tilde{Z}_n(\nu),$$
$$XY=YX,\quad XZ_n=(Z_n-1)X,\quad YZ_n=(Z_n+1)Y.\tag{3.7}$$

§ 4. Eigenfunctions. Eigenfunctions of Z_n are given by Bessel functions $J_\nu(z)$ and Neumann functions $N_\nu(z)$.

Theorem 4.1. *Dimension of the vector space* $T\cap\{u_n\in\ker(Z_n+\gamma)\}$ *is* 2. *Its bases are given by*

$$f_n(\gamma;x,y)=(-\sqrt{x/y})^{n+\gamma}J_{n+\gamma}(\sqrt{4xy}),\tag{4.1}$$
$$g_n(\gamma;x,y)=(-\sqrt{x/y})^{n+\gamma}N_{n+\gamma}(\sqrt{4xy}).$$

We have the following relations.

$$-Xf_n(\gamma;x,y)=f_n(\gamma-1;x,y),\qquad Yf_n(\gamma;x,y)=f_n(\gamma+1;x,y),\tag{4.2}$$
$$-Xg_n(\gamma;x,y)=g_n(\gamma-1;x,y),\qquad Yg_n(\gamma;x,y)=g_n(\gamma+1;x,y).$$

Since $Y(X+1)-M=Y-1$ then it follows

Theorem 4.2. *A base of the one-dimensional vector space* $T\cap\{u_n\in\ker(X+1)\}=T\cap\{u_n\in\ker(Y-1)\}$ *is given by*

$$p=\exp(y-x).\tag{4.3}$$
$$\tilde{Z}_n(\nu)p=e^{n\nu}\exp(e^{\nu}y-e^{-\nu}x)=\sum_{j=-\infty}^{\infty}e^{-j\nu}f_n(j;x,y)\tag{4.4}$$

is a base of one-dimensional vector space

$$T\cap\{u_n\in\ker(X+e^{-\nu})\cap\ker(Y-e^{\nu})\}.$$

Remark. $u_n=(\tilde{Z}_n(\nu)\pm\tilde{Z}_n(-\nu))p$ gives 1-soliton (anti-soliton) solution of the Toda equation.

§ 5. Rational solutions. $u_n=Z_n^k p$ $(k=0,1,2,\cdots)$ give a interesting series of rational solutions of the Toda equation.

Theorem 5.1 (Rational solutions).

(5.1) $$P_{n,k}=Z_n^k p/p$$

is a polynomial in (x, y) *of order* k.

(5.2) $$\rho_n=1+XY\log P_{n,k}=P_{n+1,k}P_{n-1,k}/P_{n,k}^2,$$
$$\sigma_n=Y\log P_{n-1,k}/P_{n,k}$$

is a rational solution of the Toda equation (2.2).

§6. Hypergeometric solutions. By eigenfunction expansion we can construct various solutions of the Toda equation. If

(6.1) $$u_n=\sum_{j=0}^{\infty} a_j f_n(\gamma+\varepsilon j\,;\, x, y) \qquad (\varepsilon \text{ is an integer})$$

is convergent then it belongs to T. If we choose ε and a_j suitably then we can express u_n by hypergeometric functions with two variables of order 2 which appear in Horn's list ([1]).

Theorem 6.1 (Hypergeometric solutions). *Put* $A_n=A_n(\gamma\,;\,x)=(-x)^{n+\gamma}/\Gamma(n+\gamma+1)$.

(1) $\varepsilon=1,\ a_j=(\alpha)_j(\beta)_j/j\,!$,

(6.2) $$u_n=A_n\sum_{j,k}((\alpha)_j(\beta)_j/(n+\gamma+1)_{j+k}j\,!k\,!)(-x)^j(-xy)^k$$
$$=A_n\Xi_2(\alpha,\beta,n+\gamma+1\,;\,-x,\,-xy)={}_2F_0(\alpha,\beta\,;\,Y)f_n(\gamma\,;\,x,y),$$

(6.3) $$u_n(x,0)=A_n\,{}_2F_1(\alpha,\beta,n+\gamma+1\,;\,-x),$$

(2) $\varepsilon=1,\ a_j=(\beta)_j/j\,!$,

(6.4) $$u_n=A_n\Phi_3(\beta,n+\gamma+1\,;\,-x,\,-xy)={}_1F_0(\beta\,;\,Y)f_n(\gamma\,;\,x,y),$$

(6.5) $$u_n(x,0)=A_n\,{}_1F_1(\beta,n+\gamma+1\,;\,-x),$$

(3) $\varepsilon=-1,\ a_j=(\beta)_j/(\delta)_j j\,!$,

(6.6) $$u_n=A_nH_3(-\gamma-n,\beta,\delta\,;\,x^{-1},xy)={}_1F_1(\beta,\delta\,;\,-X)f_n(\gamma\,;\,x,y),$$

(6.7) $$u_n(x,0)=A_n\,{}_2F_1(-\gamma-n,\beta,\delta\,;\,x^{-1}),$$

if we choose $\beta=\delta$ *then*

(6.8) $$u_n=A_n\exp(-2\sqrt{-xy})H_8(-2\gamma-2n,\,n+\gamma+1/2\,;\,-(4x)^{-1},\,-4\sqrt{-xy})$$
$$=\exp(-X)f_n(\gamma\,;\,x,y)=f_n(\gamma\,;\,x-1,y),$$

(4) $\varepsilon=-1,\ a_j=1/(\delta)_j j\,!$,

(6.9) $$u_n=A_nH_5(-\gamma-n,\delta\,;\,x^{-1},xy)={}_0F_1(\delta\,;\,-X)f_n(\gamma\,;\,x,y),$$

(6.10) $$u_n(x,0)=A_n\,{}_1F_1(-\gamma-n,\delta\,;\,x^{-1}),$$

(5) $\varepsilon=-2,\ a_j=1/(\delta)_j j\,!$,

(6.11) $$u_n=A_nH_{10}(-\gamma-n,\delta\,;\,x^{-2},xy)={}_0F_1(\delta\,;\,X^2)f_n(\gamma\,;\,x,y),$$

(6.12) $$u_n(x,0)=A_n\,{}_2F_1(-(\gamma+n)/2,(1-\gamma-n)/2,\delta\,;\,4x^{-2}).$$

Above series are all convergent on a suitable domain in the complex (x, y) *space.*

We used Pochhammer's notation

$$_pF_q(\alpha_1,\cdots,\alpha_p,\beta_1,\cdots,\beta_q\,;\,z)=\sum_{j=0}^{\infty}((\alpha_1)_j\cdots(\alpha_p)_j/(\beta_1)_j\cdots(\beta_q)_j j\,!)z^j,$$
$$(a)_j=\Gamma(j+a)/\Gamma(a).$$

Reference

[1] A. Erdelyi *et al.*: Higher Transcendental Functions. vol. 1, pp. 224–227, McGraw-Hill (1953).