18. Cohomology mod p of the 4-Connective Fibre Space of the Classifying Space of Classical Lie Groups

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§1. Introduction. Let G be a compact, connected, simply connected, simple Lie group. It is well known $\pi_2(G)=0$ and $\pi_3(G)=Z$. Therefore BG, the classifying space of G, is 3-connected and

$$\pi_4(BG)\cong H_4(BG)\cong H^4(BG)\cong Z.$$

Represent a generator x_4 of $H^4(BG)$ by a map $\sigma: BG\to K(Z,4)$ and denote its homotopy fibre by $B\tilde{G}$. Let p be an odd prime and denote the sequence $(p^{k-1},\cdots,p,1)$ by $I(k)$. As is well known

$$H^*(K(Z,3); Z/p)\cong Z/p[\beta\mathcal{P}^{I(k)}u_3; k\geq 1]\otimes\Lambda(\mathcal{P}^{I(k)}u_3; k\geq 0)$$

where u_3 is a generator of $H^3(K(Z,3); Z/p)\cong Z/p$. The purpose of this paper is to determine $H^*(B\tilde{G}; Z/p)$ for any classical type G. The result is

Theorem 1.1. *For any classical type G, there exists an integer $h=h(G,p)$ such that as an algebra*

$$H^*(B\tilde{G}; Z/p)\cong H^*(BG; Z/p)/(x_4,\mathcal{P}^{I(1)}x_4,\cdots,\mathcal{P}^{I(h-1)}x_4)\otimes R_h,$$

where R_h is a subalgebra of $H^*(K(Z,3); Z/p)$ generated by $\{\beta\mathcal{P}^{I(k)}u_3; k\geq 1\}\cup\{\mathcal{P}^{I(k)}u_3; k\geq h\}$. (For $h(G,p)$ see §5.)

The mod 2 cohomology of $B\tilde{G}$ for $G=SU(n)$ or $Sp(n)$ is determined in §4.

§2. Some algebraic preparations. Let V be an n-dimensional vector space over F_p. Consider a quadratic form $Q(x)$ on V. It can be thought as an element of degree 2 in $S(V^*)$, the symmetric algebra of the dual space of V. Let $B(x,y)$ be the associated bilinear form of Q (cf. Chap. 4, 1.1 of [5]) and let h be the codimension of the maximal dimensional Q-isotropic subspace of V (cf. Chap. 4, 1.3 of [5]).

Theorem 2.1. *The sequence*

$$(*)\qquad Q(x),\ B(x,x^p),\cdots,B(x,x^{p^{h-1}})$$

is a regular sequence in $S(V^*)$.

For the proof of the above theorem, we look at $\mathrm{Var}\,J$, the algebraic variety defined by J in $V\otimes\Omega$, where J is the ideal of $S(V^*)$ generated by (*) and Ω is an algebraically closed extension of F_p of infinite transcendence degree. In fact

$$\mathrm{Var}\,J=\cup\, W\otimes\Omega$$

where W ranges all maximal Q-isotropic subspaces. Theorem 2.1

follows from this fact and Proposition 1.2 of [4] (cf. [6]). Moreover we can determine the multiplicity of each primary component and the primary decomposition of J by the theorem of Macaulay and the theorem of Bezout. For any subspace W of V, $W^{\perp}$ denotes the anihilator subspace of W. Let Q_W or B_W be the restriction of Q or B to $W^{\perp}$. Denote the ideal of $S(W^{\perp *})$ generated by

$$\{Q_W(x), B_W(x, x^p), \cdots, B_W(x, x^{p^{h'}})\}$$

by $J(W^{\perp})$ where $h' = \dim W^{\perp} - \dim W$. Consider the natural map $r_W : S(V^*) \to S(W^{\perp *})/J(W^{\perp})$ and put $q_W = \operatorname{Ker} r_W$.

Theorem 2.2. *The primary decomposition of J is given by*

$$J = \cap q_W$$

where W ranges all maximal Q-isotropic subspaces.

By an easy computation we have

$$B_W(x, x^{p^{h'}}) \in J(W^{\perp}).$$

Clearly $r_W(B(x, x^{p^h})) = 0$ for any maximal Q isotropic subspace W and so we have

Lemma 2.3. *The element $B(x, x^{p^h})$ is contained in J.*

As a corollary of Theorem 2.1 and Lemma 2.3, we have the following:

Corollary 2.4. *If R is a subalgebra of $S^*(V)$ over which $S(V^*)$ is a free module and $Q(x), B(x, x^{p^k}) \in R$ for any k, then the sequence*

$$(*) \qquad Q(x),\ B(x, x^p), \cdots, B(x, x^{p^{h-1}})$$

is a regular sequence in R and $B(x, x^{p^h}) \in J'$, where J' is the ideal generated by (*) in R.

§3. Proof of Theorem 1.1. In this section p is an odd prime and $H^*(\ )$ is the mod p cohomology. The fibering $B\tilde{G} \to BG \to K(Z, 4)$ induces a fibering

$$(3.1) \qquad K(Z, 3) \longrightarrow B\tilde{G} \longrightarrow BG.$$

Pulling back (3.1) to BT we have a commutative diagram

$$\begin{array}{ccccc} K(Z,3) & \longrightarrow & B\tilde{G} & \longrightarrow & BG \\ \| & & \uparrow & & \uparrow i \\ K(Z,3) & \longrightarrow & B\tilde{T} & \longrightarrow & BT, \end{array}$$

where T is a maximal torus of G. As is well known $H^*(BT) \simeq S(V^*)$ where $V = H_2(BT) \cong (Z/p)^l$ ($l = \operatorname{rank} G$). Clearly u_3 is transgressive with $\tau(u_3) = Q(x)$ for some quadratic form Q on V. By an easy computation we have $\tau(\beta\mathcal{P}^{I(k)}u_3) = 0$ and $\tau(\mathcal{P}^{I(k)}u_3) = 2^k B(x, x^{p^k})$ $k \geq 1$. If G is classical, i^* is a monomorphism and $H^*(BT)$ is a free module over $\operatorname{Im} i^*$ (cf. [2]). Now the Serre spectral sequence for the fibering (3.1) can easily be computed by Corollary 2.4. In fact

$$E_{2p^h+3} \cong E_{\infty} \cong H^*(BG)/(x_4, \mathcal{P}^{I(1)}x_4, \cdots, \mathcal{P}^{I(h-1)}x_4) \otimes R_h.$$

The proof of $E_{\infty} \cong H^*(B\tilde{G})$ is easy.

§4. Cohomology mod 2 of $B\tilde{G}$. In this section $H^*(\ )$ is the mod 2 cohomology. Denote the sequence $(2^k, 2^{k-1}, \cdots, 2)$ by $I'(k)$. The mod 2 cohomology of $K(Z, 3)$ is isomorphic to $Z/2[Sq^{I'(k)}u_3;\ k \geq 0]$ $(Sq^{I'(0)}u_3 = u_3)$. Using the result of Quillen [4], we prove the following by a quite similar method:

Theorem 4.1. *As an algebra*

$$H^*(BS\tilde{U}(n)) \cong Z/2[c_2, \cdots, c_n]/(c_2, Sq^{I'(1)}c_2, \cdots, Sq^{I'(h-1)}c_2) \otimes R'_h$$

where R'_h is the subalgebra of $H^*(K(Z,3))$ generated by $\{(Sq^{I'(k)}u_3)^2\,;\ k<h\} \cup \{Sq^{I'(k)}u_3\,;\ k \geq h\}$ and 2^h is the Radon-Hurewiez number (see [4]).

The case $G = \mathrm{Sp}(n)$ is easy since $Sq^{I'(k)}p_1 = 0$ for $k \geq 1$ where $p_1 \in H^4(B\,\mathrm{Sp}(n))$ is a generator. Therefore we have

Theorem 4.2. *As an algebra*

$$H^*(B\,\widetilde{\mathrm{Sp}}(n)) \cong Z/2[p_2, p_3, \cdots, p_n] \otimes R'_0.$$

The case $G = \mathrm{Spin}(n)$ seems to be difficult since $H^*(BT^{[n/2]})$ is not a free module over $H^*(BG)$.

§5. The number $h(G, p)$.

For an integer n and an odd prime p, define $e(n, p)$ by

$$e(n,p) = \begin{cases} 1 & \text{if } n \equiv 2,\ p \equiv -1 \bmod 4 \\ 0 & \text{others} \end{cases}$$

and $a(n, p)$ by

$$a(n,p) = \begin{cases} n/2 + e(n,p) & \text{if } n \equiv 0 \bmod 2 \text{ and } \left(\frac{n+1}{p}\right) = 1 \\ [n/2] + 1 - e(n,p) & \text{others.} \end{cases}$$

Using the classification of quadratic forms over F_p (cf. Serre [5]), we have the following:

Theorem 5.1. (1) *If $G = SU(n+1)$, then*

$$h(G,p) = \begin{cases} a(n,p) & \text{if } \left(\frac{n+1}{p}\right) \neq 0 \\ a(n-1,p) & \text{if } \left(\frac{n+1}{p}\right) = 0. \end{cases}$$

(2) *If $G = \mathrm{Sp}(n)$, $\mathrm{Spin}(2n)$ or $\mathrm{Spin}(2n+1)$, then*

$$h(G,p) = [(n+1)/2] + e(n,p).$$

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