16. 4-Dimensional Brownian Motion is Recurrent with Positive Capacity

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1. In his pioneering work [1], Fukushima has proved that many sample path properties of the Brownian motion hold except not only on a set of the Wiener measure zero but also on a polar set with respect to the Ornstein-Uhlenbeck process on the Wiener space. Among many results, he proved that for d-dimensional Brownian motion if $d\geq 5$, then the sample paths are transient quasi everywhere, that is, except on a polar set or equivalently except on a set of capacity zero. After him, the author proved as a special case that if $d\leq 3$, the sample paths are recurrent with positive capacity or equivalently the Ornstein-Uhlenbeck process on the Wiener space hit the set of recurrent Brownian paths with positive probability (actually probability 1) [3]. In this paper, we prove that 4-dimensional Brownian paths are also recurrent with positive capacity by taking account of the result of Orey-Pruitt about the N-parameter Wiener process [4].

2. Let $W^{(2,d)}=(W_1^{(2)},\cdots,W_d^{(2)})$ be the 2-parameter Wiener process with values in d-dimensional Euclidean space R^d whose components are independent, that is, each $W_i^{(2)}$, $i=1,\cdots,d$ is an independent copy of a two parameter Gaussian process $\{W(t,s,\omega)\,;\,0\leq t,s<+\infty\}$ defined on a probability space $(\Omega,\mathcal{F},P)$ having continuous sample paths with the mean zero and the covariance

$$E[W(t_1,s_1)W(t_2,s_2)]=(t_1\wedge t_2)(s_1\wedge s_2),$$

where $a\wedge b=\min(a,b)$.

Taking $t\geq 0$ as a parameter set, $W^{(2,d)}(t,\cdot,\omega)\equiv B(t,\omega)$ is considered as a Brownian motion in the sense of Gross [2] with the values in the d-dimensional Wiener space which is a separable Banach space X with a suitable norm. We define the Ornstein-Uhlenbeck process $U(t,\omega)$ as a time change of the Brownian motion by

$$U(t,\omega)=e^{-t/2}B(e^t,\omega). \tag{1}$$

Since X is a subspace of the R^d-valued continuous functions defined on $[0,\infty)$, we denote by $f_s(x)$ for an element x of X the value in R^d at $s\geq 0$.

3. Set

$$A(u,\varepsilon)=\{x\in X\,;\,{}^{\exists}s_n\uparrow+\infty \text{ such that } \|f_{s_n}(x)-u\|<\varepsilon\},$$

where $\varepsilon>0$ and $u\in R^d$, $\|\ \|$ means the usual Euclidean norm in R^d.

Then, we have

Theorem. *If* $d \leq 4$,

$$P\{\omega\,;\,{}^{\exists}t>0 \text{ such that } U(t,\omega)\in A(u,\varepsilon)\}=1 \text{ for all } u\in R^d$$

and $\varepsilon>0$.

To prove our theorem, we need the following lemma.

Lemma. *Let I be a closed interval on* $(0,\infty)$ *and set*

$$A(I)=\{\omega\,;\,{}^{\exists}(t_n,s_n)\in I\times(0,\infty) \text{ such that } s_n\uparrow+\infty \text{ and } \|f_{s_n}(U(t_n,\omega))-u\|<\varepsilon\}.$$

If for any closed intervals on $(0,\infty)$ *we have*

$$P(A(I))=1, \tag{2}$$

then it follows that

$$P\{\omega\,;\,{}^{\exists}t>0 \text{ such that } U(t,\omega)\in A(u,\varepsilon)\}=1.$$

Proof. Let $\mathcal{T}$ be the set of all closed intervals on $(0,\infty)$ with rational end points. Then, by (2) we have

$$P(\bigcap_{I\in\mathcal{T}} A(I))=1.$$

Now take an $\omega\in\bigcap_{I\in\mathcal{T}}A(I)$ and $I_0\in\mathcal{T}$. Then, $\omega\in A(I_0)$ implies that there exists a sequence $(t_n^{(0)},s_n^{(0)})\in I_0\times(0,\infty)$ such that $s_n^{(0)}\uparrow+\infty$ and

$$\|f_{s_n^{(0)}}(U(t_n^{(0)},\omega))-u\|<\varepsilon. \tag{3}$$

Since $f_s(U(t,\omega))$ is a continuous function from $[0,\infty)\times[0,\infty)$ to R^d, there exists an interval $I_1\in\mathcal{T}$ such that

$$t_0^{(0)}\in I_1\subset I_0\cap\{t\,;\,\|f_{s_0^{(0)}}(U(t,\omega))-u\|<\varepsilon\}.$$

Again $\omega\in A(I_1)$ implies that there exists a sequence $(t_n^{(1)},s_n^{(1)})\in I_1\times(0,\infty)$ such that $s_0^{(0)}<s_n^{(1)}\uparrow+\infty$ and

$$\|f_{s_n^{(1)}}(U(t_n^{(1)},\omega))-u\|<\varepsilon.$$

Therefore, the same reason as before, there exists an interval $I_2\in\mathcal{T}$ such that

$$t_1^{(1)}\in I_2\subset I_1\cap\{t\,;\,\|f_{s_1^{(1)}}(U(t,\omega))-u\|<\varepsilon\}$$

continuing this procedure, we have a sequence of closed non empty intervals $I_1\supset I_2\supset\cdots$ and $s_n\equiv s_n^{(n)}\uparrow+\infty$ such that

$$I_{n+1}\subset I_n\cap\{t\,;\,\|f_{s_n}(U(t,\omega))-u\|<\varepsilon\}.$$

So, there exists a point $t\in\bigcap_{n=1}^{\infty}I_n$ which satisfies $\|f_{s_n}(U(t,\omega))-u\|<\varepsilon$. This implies $U(t,\omega)\in A(u,\varepsilon)$.

Proof of Theorem. It is sufficient to check the condition of the lemma. But in their paper [4], Orey and Pruitt have proved that

$$P\{\omega\,;\,{}^{\exists}(t_n,s_n)\in[1,2]\times(0,\infty) \text{ such that } s_n\uparrow+\infty \;\; \|W^{(2,4)}(t_n,s_n,\omega)\|\leq 1\}=1.$$

It is easy to show that their proof works for recurrency of any open set in R^d. By the relation (1), this is also true for the Ornstein-Uhlenbeck process. This completes the proof of the theorem.

References

[1] M. Fukushima: Basic properties of Brownian motion and a capacity on the Wiener space. J. Math. Soc. Japan, **36**, 161–175 (1984).

[2] L. Gross: Potential theory on Hilbert space. J. Funct. Anal., **1**, 123–181 (1967).

[3] N. Kôno: Propriétés quasi-partout de fonctions aléatoires gaussiennes. Lecture Note of the University of Strasbourg (1983).

[4] S. Orey and W. E. Pruitt: Sample functions of the N-parameter Wiener process. Ann. of Prob., 138–163 (1973).