15. *On the Essential Spectrum of MHD Plasma in Toroidal Region*

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1. Introduction. Related to the plasma confinement problem, the following second order differential equation is of some interest:

(1.1) $$\rho(\partial^2\xi/\partial t^2)=\operatorname{grad}\{\gamma P(\operatorname{div}\xi)+(\operatorname{grad}P)\cdot\xi\}$$
$$+(1/\mu)\{B\times\operatorname{rot}(\operatorname{rot}(B\times\xi))-(\operatorname{rot}B)\times\operatorname{rot}(B\times\xi)\}$$
$$\equiv-\rho^{1/2}K\rho^{1/2}\xi.$$

Here, $\xi(t,r)$ is related to the velocity field $V(t,r)$ as $d\xi/dt=V(t,\xi+r)$, $\xi(0,r)=0$, and is called the Lagrangian displacement vector. The quantities ρ, P and B are independent of t and are the solutions of the plasma equilibrium satisfying:

(1.2) $$\operatorname{grad}P=j\times B,\quad j=(1/\mu)\operatorname{rot}B,\quad \operatorname{div}B=0,$$

with P, $\rho\geqq\varepsilon_0>0$. Further, (1.1) is derived from the following magnetohydrodynamic (MHD in short) system:

(1.3) $$\begin{cases}\dfrac{\partial\rho}{\partial t}+\operatorname{div}(\rho V)=0, & \dfrac{D}{Dt}(P\rho^{-\gamma})=0,\quad \rho\dfrac{DV}{Dt}=-\operatorname{grad}P+j\times B,\\ \dfrac{\partial B}{\partial t}=-\operatorname{rot}E, & \operatorname{div}B=0,\quad E+V\times B=0,\quad j=\dfrac{1}{\mu}\operatorname{rot}B,\end{cases}$$

by means of the linearization in the vicinity of the equilibrium (1.2). Here, ρ, P, V and j are respectively the density, the pressure, the velocity and the electric current density of the plasma, and B and E are the magnetic and electric fields, and μ is the permeability and γ is the specific heat ratio, and $D/Dt=\partial/\partial t+V\cdot\operatorname{grad}$ is the convective derivative.

In the following, we shall investigate the spectral properties of K. Especially, we consider (1.1) in the axisymmetric toroidal region Ω in R^3 and around the following special axisymmetric equilibrium (cf. Temam [5], Friedman [2] §§ 14–18). Namely, Ω is defined as

$$\Omega=\{r=(x,y,z)\,|\,a_1<\psi(r,\vartheta,z)<a_2,\ x=r\cos\vartheta,\ y=r\sin\vartheta\},$$

where $\psi=\psi(r,z)$ with $r=(x^2+y^2)^{1/2}$ satisfies the non-linear elliptic differential equation (Grad-Shafranov equation):

$$-\left(r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\psi=r^2\{\partial P/\partial\psi\}+I\{\partial I/\partial\psi\}$$

with given functions P and I of ψ. In this case, B is given as

$$B=\left(\frac{1}{r}\frac{\partial\psi}{\partial z},\ \frac{1}{r}I,\ -\frac{1}{r}\frac{\partial\psi}{\partial r}\right).$$

We shall assume that we can take the orthogonal coordinates (ψ, χ, ϑ) with $\chi=\chi(r, z)$, $0\leqq\chi<2\pi$, satisfying: grad $\psi\cdot$grad $\chi=0$.

Then, for the displacements $\xi=e^{in\vartheta}\eta(\psi, \chi)$, the operator K in (1.1) is represented in the form:

(1.4) $$K=\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where

(1.5) $$\begin{aligned} -A&=\partial_\psi\alpha_{11}\partial_\psi+\partial_\chi\alpha_{22}\partial_\chi+\partial_\chi\alpha_2-\alpha_2^*\partial_\chi-\alpha_0 \\ -B&=\partial_\psi(\beta_{12}\partial_\chi+\beta_1)+\partial_\chi\beta_2+\beta_0 \\ -B^*&=(-\partial_\chi\beta_{12}^*+\beta_1^*)(-\partial_\psi)-\beta_2^*\partial_\chi+\beta_0^* \\ -C&=\partial_\chi\gamma_{22}\partial_\chi+\partial_\chi\gamma_2-\gamma_2^*\partial_\chi-\gamma_0 \end{aligned}$$

with $\partial_\psi=\partial/\partial\psi$, $\partial_\chi=\partial/\partial\chi$ (cf. Goedbloed [3]). Here, α_{11}, α_{22} and α_0 are real functions, and γ_{22} and γ_0 are 2×2 real symmetric matrix valued functions, and α_2 is a function, and $\beta_{12}, \beta_1, \beta_2$ and β_0 are 1×2 matrix valued functions, and γ_2 is a 2×2 matrix valued function, and they are all bounded and smooth and 2π-periodic in χ-variable together with their all derivatives. Further, we denote by * the adjoint of a matrix.

The operators A, B, B^* and C will be realized in $\mathcal{C}(\mathcal{N})$, $\mathcal{C}(\mathcal{N}\oplus\mathcal{N}, \mathcal{N})$, $\mathcal{C}(\mathcal{N}, \mathcal{N}\oplus\mathcal{N})$ and $\mathcal{C}(\mathcal{N}\oplus\mathcal{N})$ respectively, where $\mathcal{C}(\cdot)$ denotes a class of closed operators in respective spaces and $\mathcal{N}=L^2(\Omega^*)$ with a flat product manifold $\Omega^*: \Omega^*=\{(\psi, \chi)\,|\,a_1<\psi<a_2,\ 0\leqq\chi<2\pi,\ (\psi, 0)$ is identified with $(\psi, 2\pi)\}=(a_1, a_2)\times S^1$, where $S^1=\boldsymbol{R}/2\pi\boldsymbol{Z}$. Further, A is strongly elliptic in ψ and χ variables with Dirichlet boundary condition, and C is elliptic in χ variable uniformly with respect to a parameter ψ $(a_1<\psi<a_2)$, and the inequality:

(1.6) $$\{\gamma_{22}-(1/\alpha_{11})\beta_{12}^*\beta_{12}\}(\psi, \chi)\geqq c_0>0, \qquad (\psi, \chi)\in\Omega^*$$

is satisfied.

2. Selfadjoint realization of K. We shall consider the case that the plasma is confined in the fixed conducting shell. We denote by $H^m(\Omega^*)$ the Sobolev space of order m in Ω^*, and by $H_\circ^m(\Omega^*)$ a subset of the functions in $H^m(\Omega^*)$ with zero trace at the boundary of Ω^* up to the $(m-1)$-th order derivatives. Let

$$\mathcal{D}(K_\circ)=(H_\circ^1(\Omega^*)\cap H^2(\Omega^*))\oplus\{H^2(\Omega^*)\oplus H^2(\Omega^*)\},$$

and let

$$K_\circ=\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

with domain $\mathcal{D}(K_\circ)$. Then, $K_\circ$ is symmetric in $\mathcal{H}=\mathcal{N}\oplus\{\mathcal{N}\oplus\mathcal{N}\}$ and we have;

Theorem 1. *Under the conditions in §1, the above $K_\circ$ has a selfadjoint extension K in $\mathcal{H}$ which has a resolvent for a sufficiently large λ with the form:*

(2.1) $R_\lambda=(K+\lambda)^{-1}$

$$=\begin{pmatrix}(A+\lambda)^{-1}+[(A+\lambda)^{-1}B]D_\lambda^{-1}B^*(A+\lambda)^{-1} & -[(A+\lambda)^{-1}B]D_\lambda^{-1}\\ -D_\lambda^{-1}B^*(A+\lambda)^{-1} & D_\lambda^{-1}\end{pmatrix}$$

with $D_\lambda=[-B^*(A+\lambda)^{-1}B+C+\lambda]$. *Here,* $[\cdot]$ *denotes a closed extension of an operator.*

We can prove this theorem by introducing the symmetric closed form which corresponds to D_λ and has the form domain $\mathcal{D}=\mathcal{D}(D_\lambda^{1/2})=\{L^2([a_1,a_2])\otimes H^1(S^1)\}^2$ which includes the domain $\mathcal{D}(D_\lambda)$ of D_λ. Here, $H^1(S^1)$ denotes the Sobolev space of order 1 in S^1.

3. Essential spectrum of K. The essential spectrum of K was first investigated extensively by J. P. Goedbloed [3] and later by J. Descloux and G. Geymonat [1] in mathematically rigorous fashion. We shall show here another proof which will be simpler than that of [1] in some sense, and it may have some advantage for further investigations of the spectrum of K such as the absolute continuity (cf. Kako [4]).

The main idea of the proof is that the essential spectrum is invariant under the perturbations by compact operators (denoted by C_∞). Namely, the resolvent R_λ of K is represented as

(3.1) $$R_\lambda=\begin{pmatrix}0 & 0\\ 0 & D_\lambda^{\circ\,-1}\end{pmatrix}+R_\lambda^\infty,\ R_\lambda^\infty\in C_\infty(\mathcal{H})$$

where

(3.2) $$D_\lambda^\circ(=D_\lambda^\circ(\psi))=-\partial_\chi\left\{\gamma_{22}-\frac{\beta_{12}^*\beta_{12}}{\alpha_{11}}\right\}\partial_\chi+\left\{\gamma_2-\frac{\beta_1^*\beta_{12}}{\alpha_{11}}\right\}\partial_\chi -\partial_\chi\left\{-\frac{\beta_{12}^*\beta_1}{\alpha_{11}}+\gamma_2^*\right\}+\left\{\gamma_0-\frac{\beta_1^*\beta_1}{\alpha}\right\}+\lambda$$

(freezing operator)

and $D^\circ\equiv D_0^\circ$ has the essential spectrum $\Sigma=\bigcup_j\Sigma_j$ with $\Sigma_j=\{\lambda_j(\psi)\,|\,a_1\leqq\psi\leqq a_2$, $\lambda_j(\psi)$ is the j-th eigenvalue of $D_\lambda^\circ(\psi)$ acting on the variable χ with a fixed $\psi\}$. Now, the main theorem is stated as follows.

Theorem 2. *The essential spectrum of K coincides with Σ defined above.*

4. Outline of the proof of Theorem 2. The proof of the theorem is derived from the next lemma.

Lemma 3. *The operator D_λ^{-1} is represented as*

(4.1) $$D_\lambda^{-1}=D_\lambda^{\circ\,-1}-D_\lambda^{-1}Q_\lambda D_\lambda^{\circ\,-1},\qquad Q_\lambda\equiv D_\lambda-D_\lambda^\circ,$$

with $D_\lambda^{-1}Q_\lambda D_\lambda^{\circ\,-1}\in C_\infty(\mathfrak{N}\otimes\mathfrak{N})$.

Proof. After some calculations, we have

(4.2) $$Q_\lambda=D_\lambda-D_\lambda^\circ=-\partial_\chi\beta_{12}^*(A_1+\lambda)^{-1}(-\partial_\chi\alpha_{22}\partial_\chi+\lambda)\alpha_{11}^{-1}\beta_{12}\partial_\chi+Q_\lambda^\infty$$

with some second order elliptic operator A_1 in ψ and χ and the lower order term Q_λ^∞ which is compact in $\mathfrak{N}\oplus\mathfrak{N}$. Then, we have

(4.3) $$D_\lambda^{-1}Q_\lambda D_\lambda^{\circ\,-1}=\{D_\lambda^{-1}(-\partial_\chi\beta_{12}^*)\}\{(A_1+\lambda)^{-1}(-\partial_\chi\alpha_{22}\partial_\chi+\lambda)^{1/2}\}\times\{(-\partial_\chi\alpha_{22}\partial_\chi+\lambda)^{1/2}\alpha_{11}^{-1}(\beta_{12}\partial_\chi)D_\lambda^{\circ\,-1}\}+\text{compact operator},$$

and the first and the third factors of the first term are bounded since $\mathscr{D}(D_\lambda)\subset\mathscr{D}(D_\lambda^{1/2})=\{L^2([a_1, a_2])\otimes H^1(S^1)\}^2$ and the second factor

$$(A_1+\lambda)^{-1}(-\partial_\chi\alpha_{22}\partial_\chi+\lambda)^{1/2}$$

is compact. Hence $D_\lambda^{-1}Q_\lambda D_\lambda^{\circ -1}$ becomes compact. Q.E.D.

References

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