13. *Stationary Solutions of a Spatially Aggregating Population Model*

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We are concerned with stationary solutions of the following nonlinear degenerate diffusion equation

$$(1)\qquad u_t=(u^m)_{xx}+\left[u\left(\int_{x-r}^{x} u\,dy-\int_{x}^{x+r} u\,dy\right)\right]_x \qquad \text{in } \boldsymbol{R}\times(0,\infty)$$

where $m>1$ is a constant, $0\leqq r\leqq\infty$ a parameter and $u(x,t)\geqq 0$ denotes the population density at position $x\in\boldsymbol{R}$ and time $t>0$. Equations of this type, proposed by Nagai and Mimura [4], represent a spatially spreading population model for a class of aggregating phenomena of individuals. The first term corresponds to the transport of population through a nonlinear diffusion process called density-dependent dispersal (e.g. Gurney and Nisbet [2], Gurtin and MacCamy [3]). The second term provides an aggregative mechanism that moves individuals to the right (resp. left) direction when

$$\int_{x}^{x+r} u(y,t)\,dy>\int_{x-r}^{x} u(y,t)\,dy \qquad (\text{resp. } <).$$

Thus a non-trivial stationary solution of (1) ecologically exhibits an aggregation of individuals.

For a class of Cauchy problems including (1) subject to a nonnegative initial condition $u(x,0)=u^0(x)\geqq 0$ for $x\in\boldsymbol{R}$, Nagai [6] has shown the existence and uniqueness of weak solution. He has also obtained some properties of the solution, for instance, the finite speed of propagation of initial disturbance.

In the cases of $r=0$ and $r=\infty$, stationary solutions of the equation (1) have already been obtained. When $r=0$, (1) is reduced to the porous medium equation (e.g. Aronson [1]) which has no non-trivial stationary solution. In the case of $r=\infty$, Nagai and Mimura [5] have shown that (1) has non-trivial stationary solitary wave solutions.

In the present paper, we deal with the case of $0<r<\infty$, restricting $m=2$. A stationary solution $u(x)$ of (1) with $m=2$ is defined to be a non-negative function belonging to $C(\boldsymbol{R})\cap L_1(\boldsymbol{R})$ that satisfies

(i) $u^2\in C^1(\boldsymbol{R})$,

(ii) $(u^2)_x+u\left(\int_{x-r}^{x} u\,dy-\int_{x}^{x+r} u\,dy\right)=0.$

The main result is described as:

Theorem. *The equation* (1) *has no non-trivial stationary solution if* $r \leqq \sqrt{2}$. *If* $r > \sqrt{2}$, (1) *has stationary pulse-like solutions characterized as below.*

We introduce a function space $H_{00}^{1,s}$ defined by

$$H_{00}^{1,s} = \{ f \in H^1(\boldsymbol{R})\,;\ f(x)=0 \text{ for } |x| \geqq \pi/2,\ f(x)=f(-x) \text{ for all } x \in \boldsymbol{R} \},$$

and associate a symmetric bilinear form $L_{r,b} : H_{00}^{1,s} \times H_{00}^{1,s} \to \boldsymbol{R}$ given by

$$L_{r,b}(f, g) = \int_{-\infty}^{\infty} \left[\frac{df}{dx}\frac{dg}{dx} - \frac{r^2}{2}\,\frac{f(x)-f(x-b)}{b}\,\frac{g(x)-g(x-b)}{b} \right] dx$$

with $r>0$ and $b>0$. For the eigenvalue problem

$$\begin{cases} \text{Find } \lambda \in \boldsymbol{R} \text{ and } f \in H_{00}^{1,s} \text{ such that} \\ L_{r,b}(f, g) = \lambda \int_{-\infty}^{\infty} fg\,dx \qquad \text{for all } g \in H_{00}^{1,s}, \end{cases}$$

we have obtained:

Proposition. *There exists an increasing sequence of eigenvalues of* $L_{r,b}$:

$$\lambda_1(r, b) \leqq \lambda_2(r, b) \leqq \cdots \qquad \textit{with } \lim_{k\to\infty} \lambda_k(r, b) = \infty.$$

If $r \leqq \sqrt{2}$, *the principal eigenvalue* $\lambda_1(r, b)$ *is strictly positive for all* $b>0$. *For each* $r>\sqrt{2}$, *there exists a unique* $b=b(r)>0$ *such that the principal eigenvalue* $\lambda_1(r, b)=0$. *This principal eigenvalue is simple and the principal eigenfunction* $f_r(x)$ *is of one sign on the interval* $(-\pi/2, \pi/2)$. (We may assume that $f_r(x)$ is positive on $(-\pi/2, \pi/2)$ and $\int_{-\infty}^{\infty} |f_r|^2\,dx = 1$.) *Moreover, it holds that*

(a) $b=b(r)$ *is continuous and strictly increasing on the interval* $(\sqrt{2}, \infty)$,

(b) $b(r) \downarrow 0$ *as* $r \downarrow \sqrt{2}$ *and* $b(r)=r$ *for* $r \geqq \pi$,

(c) $f_r(x)$ *is a unimodal function.*

We now give a formula for the non-trivial stationary solution of (1) with $m=2$. Using the above results, we let

$$d(r) = r\pi/2b(r) \quad \text{and} \quad u_r(x) = f_r(b(r)x/r).$$

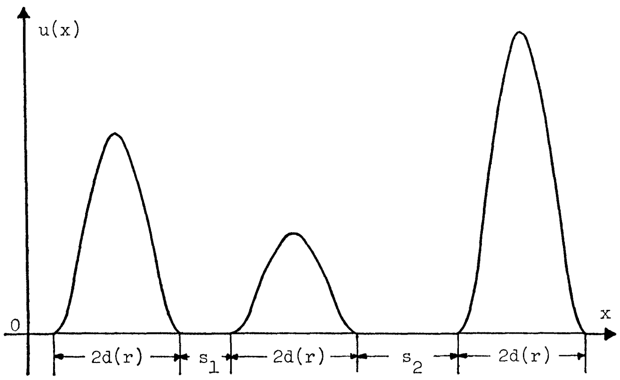

Fig. 1. A stationary solution of (1) ($s_1 \geqq r$, $s_2 \geqq r$).

Then, all the non-trivial stationary solutions $u(x)$ of (1) are represented in the form

$$u(x)=\sum_{i=M}^{N} c_i u_r(x-a_i)$$

(Fig. 1), where $\{c_i\}_{i=M}^{N}$ and $\{a_i\}_{i=M}^{N}$ are finite or infinite sequences of real numbers (i.e., $-\infty \leqq M < \infty$, $-\infty < N \leqq \infty$, $M \leqq N$) that satisfy the conditions

$$c_i>0, \quad a_i \leqq a_{i+1}-2d(r)-r, \quad \sum_{i=M}^{N} c_i<\infty.$$

Remark. In the case of $r=\infty$, Nagai and Mimura [5] have obtained the asymptotic behavior of a solution of the Cauchy problem (1) subject to a non-negative initial condition: It tends to a stationary solution uniquely determined by the initial condition as time t tends to infinity (Fig. 2).

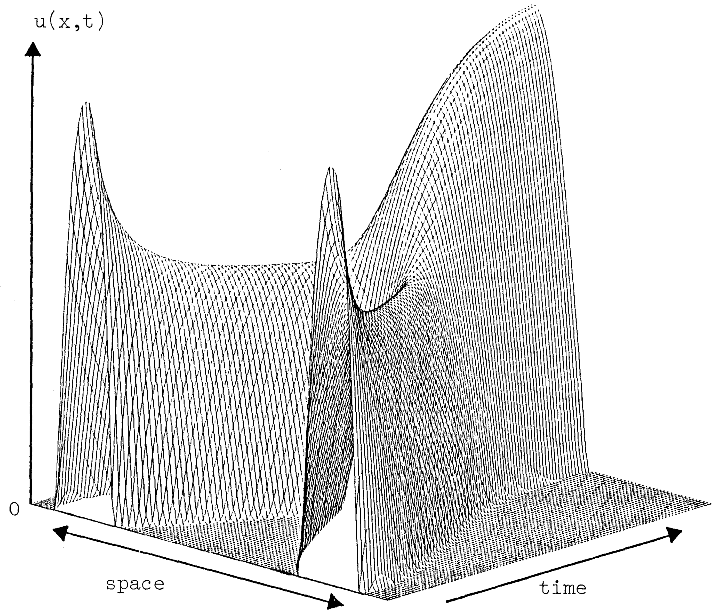

Fig. 2. A solution of (1) subject to a non-negative initial condition.

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