107. *Links of Homeomorphisms of Surfaces and Topological Entropy*

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1. Introduction. In [5] the author defined a link L from a given orientation preserving homeomorphism g of a 2-disk D^2 and a finite number of periodic orbits of g in $\mathrm{Int}\,D^2$, and showed that the topological entropy $h(g)$ of g is closely related to the link type of L. In this paper, we extend this result to orientation preserving homeomorphisms of general compact, orientable surfaces. Moreover, we will prove deeper results on periods of periodic orbits of the maps.

In this paper, we suppose that surfaces are connected.

Let F be a compact, orientable surface, f an orientation preserving self-homeomorphism of F and Σ a set consisting of a finite number of periodic orbits of f in $\mathrm{Int}\,F$ with $\chi(F-\Sigma)<0$. Let M_f be the mapping torus of f, i.e. M_f is obtained from $F\times[0,1]$ by identifying $(x, 0)$ to $(f(x), 1)$ $(x \in F)$. Then $\Sigma\times[0,1]$ projects to a link $L_{f,\Sigma}$ in M_f. Then our result is:

Theorem. *If* $h(f)=0$, *then* $L_{f,\Sigma}$ *is a graph link. Conversely, if* $L_{f,\Sigma}$ *is a graph link, then* f *is isotopic rel* Σ *to a homeomorphism* g *such that* $h(g)=0$. *Moreover, if* $L_{f,\Sigma}$ *is not a graph link and* f *is differentiable at each point of* Σ, *then* f *has infinitely many periodic orbits whose periods are mutually distinct.*

For the definition of graph link see section 2 below.

I would like to express my gratitude to Dr. Koichi Yano for suggesting me to consider this theorem.

2. Preliminaries. A general reference of *topological entropy* is [2, Exposé 10]. In [7] Thurston has shown that if ψ is a homeomorphism of a compact, hyperbolic surface F, then ψ is isotopic to ψ' which is either (1) periodic, (2) pseudo-Anosov, or (3) reducible i.e. there is a system of mutually disjoint simple loops Γ on F such that $\psi'(\Gamma)=\Gamma$; Γ has a ψ'-invariant regular neighborhood $\eta(\Gamma)$ and each ψ'-component of F-$\mathrm{Int}\,\eta(\Gamma)$ satisfies (1) or (2); each component A_i of $\eta(\Gamma)$ is mapped to itself by some positive iterate $(\psi')^{m_i}$ of ψ' and $\psi'^{m_i}|_{A_i}$ is a twist homeomorphism of an annulus [6]. We call ψ' *Thurston's cannonical form* of ψ.

The next assertion was used in the proof of Theorem of [3], but no proof was given there.

Proposition 2.1. *If ψ' contains a pseudo-Anosov component, then ψ has infinitely many periodic orbits whose periods are mutually distinct.*

Proof. If necessary, by taking ψ^2 we may suppose that ψ is orientation preserving. If ψ' is pseudo-Anosov, then it is shown that ψ' has dense periodic points and ψ'^n has $N(\psi^n)$ fixed points ([2], [7]), where $N(\psi^n)$ denotes the number of essential fixed point classes ([1]), say Nielsen number. Then, by using the theory of Markov partitions ([2 ; Exposé 10]), we can show that $N(\psi'^n)\to\infty$ (as $n\to\infty$). Since $N(\psi^n)$ is a lower bound for all maps homotopic to ψ^n, we see that the Proposition holds.

Suppose that ψ' is reducible by Γ with ψ'-invariant regular neighborhood $\eta(\Gamma)$. If necessary, by taking some power of ψ we may suppose that ψ' fixes each component of $\eta(\Gamma)$ and F-Int $\eta(\Gamma)$. Let F' be a component of F-Int $\eta(\Gamma)$ such that $\psi'|_{F'}$ is a pseudo-Anosov. By the arguments in [4] we can show that the fixed points of ψ'^n in F' represent mutually different fixed point classes. Hence, by the argument as above we see that $N(\psi^n)\to\infty$ $(n\to\infty)$, and the Proposition holds.

A *link* L is a finite union of mutually disjoint circles in a 3-manifold. The *exterior* of L is the closure of the complement of a regular neighborhood of L. A 3-manifold M is a *graph manifold* if there is a system of mutually disjoint 2-tori $\{T_i\}$ in M such that the closure of each component of M cut along $\cup T_i$ is a (surface)$\times S^1$. A link L is a *graph link* if the exterior of L is a graph manifold. Then we have:

Proposition 2.2 ([5, Proposition 2.1]). *Let f be an orientation preserving homeomorphism of a compact, orientable, hyperbolic surface. Then Thurston's cannonical form of f does not contain a pseudo-Anosov component if and only if the mapping torus M_f is a graph manifold.*

3. Proof of Theorem. Let $F, f, \Sigma, M_f, L_{f,\Sigma}$ be as in section 1. Then f is isotope rel Σ to a diffeomorphism g. Let S be a surface obtained from $F-\Sigma$ by adding a circle to each end. Since g is differentiable at each point of Σ, $g|_{F-\Sigma}$ extends to $\bar{g}\,; S\to S$ ([3]). Then the exterior of $L_{f,\Sigma}$ is homeomorphic to the mapping torus $M_{\bar g}$.

Suppose that $h(f)=0$. We can show that $h(\bar{g})=h(g)$. Assume that Thurston's cannonical form of $\bar{g}$ contains a pseudo-Anosov component. Then by [3, section 3] we see that $h(f)>0$, which is a contradiction. Then by Proposition 2.2 $M_{\bar g}$ is a graph manifold. Hence $L_{f,\Sigma}$ is a graph link.

Suppose that $L_{f,\Sigma}$ is a graph link. Then $M_{\bar g}$ is a graph manifold, and by Proposition 2.2 Thurston's cannonical form $\bar{\psi}$ of $\bar{g}$ does not contain a pseudo-Anosov component. Let $\psi; F\to F$ be the projection of $\bar{\psi}$. Then ψ is isotopic to f rel Σ and $h(\psi)=0$.

The last conclusion of Theorem follows from the fact that $f|_{F-\Sigma}$ extends to $\bar{f}: S\to S$, and Proposition 2.2 and Proposition 2.1.

This completes the proof of Theorem.

References

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