103. On Some Euler Products. II

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§1. **Meromorphy of Euler products.** Let $E=(P, G, \alpha)$ be an Euler datum in the sense of Part I. We describe a sufficient condition making E and $\bar{E}=(P, G\times \boldsymbol{R}, \bar{\alpha})$ complete when $\mu(P)<d(P)$ $(<\infty)$. We follow the notations of Part I (see [1]).

We say that E satisfies the condition L if E satisfies the following (I)–(III):

(I) $L(s, E, \rho)$ is meromorphic on $\boldsymbol{C}$ for each $\rho\in \mathrm{Irr}^u(G)$.

(II) $L(s, E, \rho)$ is non-zero holomorphic in $\mathrm{Re}(s)\geqq d(P)$ for each $\rho\in \mathrm{Irr}^u(G)$, except for a simple pole at $s=d(P)$ when ρ is trivial.

(III) For each $\rho\in \mathrm{Irr}^u(G)$ and $T>0$, let $S(T, E, \rho)$ be the number of distinct zeros and poles of $L(s, E, \rho)$ in the region $\{s\in \boldsymbol{C};\ 0<\mathrm{Re}(s)\leqq d(P)$ and $-T<\mathrm{Im}(s)<T\}$. Then there exist a positive constant c and a real valued "admissible" function C on $\mathrm{Irr}^u(G)$ such that the following holds:

$$S(T, E, \rho)<C(\rho)(T+1)^c \qquad \text{for all } \rho\in \mathrm{Irr}^u(G) \text{ and } T>0.$$

The admissibility of C is defined as follows. We denote by $\mathrm{Rep}^u(G)$ the set of all equivalence classes of finite dimensional continuous unitary representations of G, which is considered to be a free abelian semigroup (with respect to the direct sum $\oplus$) generated by $\mathrm{Irr}^u(G)$, hence C is naturally considered as a function on $\mathrm{Rep}^u(G)$ by the additive extension. We put $C_0(\rho)=C(\rho)/\deg(\rho)$. We say that C is admissible if there exists a constant $a>0$ such that C_0 satisfies the following (1)–(3):

(1) $C_0(\rho_1\otimes\rho_2)\leqq C_0(\rho_1)+C_0(\rho_2)+a$ for all ρ_1 and ρ_2 in $\mathrm{Rep}^u(G)$;

(2) $C_0(\wedge^j(\rho))\leqq C_0(\rho)j\cdot\deg(\rho)+a$ for all ρ in $\mathrm{Rep}^u(G)$ and $j\geqq 0$, where $\wedge^j(\rho)$ denotes the j-th exterior power of ρ;

(3) $C_0(S^m(\rho))\leqq C_0(\rho)m\cdot\deg(\rho)+a$ for all ρ in $\mathrm{Rep}^u(G)$ and $m\geqq 0$, where $S^m(\rho)$ denotes the m-th symmetric power of ρ.

(For example, deg is an admissible function with any $a\geqq 1$.)

Then we have the following

Theorem 1. *Let $E=(P, G, \alpha)$ be an Euler datum with $\mu(P)<d(P)$. Assume that E satisfies the condition L. Then E and $\bar{E}$ are complete.*

§2. **Note on the proof.** Let G be a topological group. Let $H(T)$ be a polynomial of degree r belonging to $1+T\cdot R^u(G)[T]$. Then, there are continuous functions $\gamma_m : \mathrm{Conj}(G)\to \boldsymbol{C}$ such that

$$H_c(T)=\prod_{m=1}^{r}(1-\gamma_m(c)T) \qquad \text{for all } c\in \mathrm{Conj}\,(G),$$

where Conj (G) is equipped with the quotient topology induced from G. (Note that Conj (G) is the quotient space of G by the inner automorphism group.) We put $\gamma(c)=\max\{|\gamma_m(c)|\,;\, m=1,\cdots,r\}$. Then γ is a real valued continuous function on Conj (G), and we define $\gamma(H)=\sup\{\gamma(c)\,;\, c\in \mathrm{Conj}\,(G)\}$. Then we have: $1\leqq\gamma(H)<\infty$. (When $H(T)=1$, we define $\gamma(H)=1$.) Moreover we see that $H(T)$ is unitary iff $\gamma(H)=1$. Then we have

Proposition 1. *Let G and $H(T)$ be as above. Then:*

(1) *There is a unique $\kappa(n,\rho)\in \boldsymbol{Z}$ for each integer $n\geqq 1$ and $\rho\in \mathrm{Irr}^u(G)$ such that $\kappa(n,\rho)=0$ except for a finite number of ρ for each fixed n and the following identity holds in the multiplicative group $1+T\cdot R^u(G)[[T]]$:*

$$H(T)=\prod_{n\geqq 1}\prod_{\rho} D_\rho(T)^{\kappa(n,\rho)}$$

where $D_\rho(T)=\det(1-\rho T)\in 1+T\cdot R^u(G)[T]$.

(2) *$|\kappa(n,\rho)|\leqq \deg(H)(d(n)/n)\gamma(H)^n$ for all n and ρ, where $d(n)$ denotes the number of the divisors of n, and* deg (H) *denotes the degree of $H(T)$.*

(3) *Put $f(n)=\sum_\rho \deg(\rho)$ where ρ runs over the finite set $I_n(H)=\{\rho\in \mathrm{Irr}^u(G)\,;\,\kappa(n,\rho)\neq 0\}$. Then, there are positive constants $c(1)$ and $c(2)$ satisfying the following: $f(n)\leqq c(1)n^{c(2)}$ for all $n\geqq 1$.*

(4) *If $c\in \mathrm{Conj}\,(G)$, $T\in \boldsymbol{C}$, and $|T|<\gamma(H)^{-1}$, then the right hand side of*

$$H_c(T)=\prod_{n\geqq 1}\prod_{\rho} D_{\rho(c)}(T^n)^{\kappa(n,\rho)}$$

converges absolutely as an infinite product.

We notice that (3) is a crucial point, and in the proof we show an explicit estimation concerning the set $I_n(H)$. This refinement is important in the proof of Theorem 1 (the part "$\tilde{E}: U\Rightarrow \bar{\tilde{E}}: D$"; see below).

Now, let $E=(P,G,\alpha)$ be an Euler datum. Let $H(T)\in 1+T\cdot R^u(G)[T]$. Then, using Proposition 1, we have the absolutely convergent expression

$$(*) \qquad L(s,E,H)=\prod_{n\geqq 1}\prod_{\rho} L(ns,E,\rho)^{\kappa(n,\rho)}$$

when $\mathrm{Re}\,(s)>\max\{d(P), (\log\gamma(H))/(\log N_1)\}$, where N_1 denotes the first (or, minimal) norm of P defined by $N_1=\min\{N(p)\,;\,p\in P\}$. Suppose that $H(T)$ is unitary (i.e., $\gamma(H)=1$). Then, by (2) of Proposition 1, there is an integer $N\geqq 1$ such that $\kappa(n,\rho)=0$ for all $n>N$ and all $\rho\in \mathrm{Irr}^u(G)$. Hence, $(*)$ is a finite product, so $L(s,E,H)$ is meromorphic on $\boldsymbol{C}$ if $L(s,E,\rho)$ are meromorphic on $\boldsymbol{C}$ for all $\rho\in\mathrm{Irr}^u(G)$. Therefore the large part of the proof of Theorem 1 treats the case of the nonunitary $H(T)$; in this case $(*)$ is actually an infinite product. To study this case, we introduce the compactification $\tilde{E}$ of E. We denote by $K(G)$ the Bohr compactification of G; let $\varphi: G \to \prod_\rho U(\deg(\rho))$ be the continuous homomorphism defined by $\varphi(g)=(\rho(g))_\rho$ where ρ runs over $\mathrm{Rep}^u(G)$ and $U(n)$ denotes the unitary group of size n, then $K(G)$ is the topological closure of $\varphi(G)$. Then, this pair $(K(G), \varphi)$ has the following universal property: let (K, ψ) be any pair of compact group K and a continuous homomorphism $\psi: G \to K$, then, there exists a unique continuous homomorphism $f: K(G) \to K$ such that $\psi = f \circ \varphi$. In particular there exists a natural bijection between $\mathrm{Irr}^u(G)$ and $\mathrm{Irr}^u(K(G))$. Now we define the compactification by $\tilde{E}=(P, K(G), \tilde{\alpha})$ where $\tilde{\alpha}=\tilde{\varphi}\circ\alpha$ with the map $\tilde{\varphi}: \mathrm{Conj}(G) \to \mathrm{Conj}(K(G))$ induced from φ. (We remark that this compactification is determined up to "isomorphism", and this ambiguity has no effect on our argument.)

We say that an Euler datum $E=(P, G, \alpha)$ is compact if G is compact. (For example, the compactification of an Euler datum is compact.) For each compact Euler datum $E=(P, G, \alpha)$ we introduce a condition U ("uniformity") which is weaker than L. For $t>0$ and a subset S of $\mathrm{Conj}(G)$ we put $\pi(t, E, S)=\#\{p \in P ; N(p) \leqq t \text{ and } \alpha(p) \in S\}$. We say that E satisfies the condition U if E satisfies (I) and (III) of L and the following:

$$\text{(II-U)} \qquad \pi(t, E, S) \sim \frac{m(S)t^{d(P)}}{d(P)\log t} \qquad \text{as } t \to \infty$$

for each subset S of $\mathrm{Conj}(G)$ such that the boundary of S has measure zero for m, where m denotes the normalized measure on $\mathrm{Conj}(G)$ induced from the normalized Haar measure on G.

The proof of Theorem 1 goes as follows:

$E: L \Leftrightarrow \tilde{E}: L \Rightarrow \tilde{E}: U \Rightarrow \bar{\tilde{E}}: C \Rightarrow \bar{E}: C\ (\Rightarrow E: C)$, where C denotes "complete". The implication $\tilde{E}: U \Rightarrow \bar{\tilde{E}}: C$ is the most essential part, and we show that $\tilde{E}: U \Rightarrow \bar{\tilde{E}}: D \Rightarrow \bar{\tilde{E}}: C$ by introducing a condition D ("density"), where the condition $\mu(P)<d(P)$ is used.

§3. **Examples.** We note two typical examples of complete Euler data. Some of other examples are automorphic (Langlands type) and schematic (Hasse-Weil type). (See also Examples 1–3 of Part I.) The first example is the Euler datum of Artin-Hecke type described in §3 of Part I. Let $E=E(\bar{F}/F)=(P(O_F), W(\bar{F}/F), \alpha)$ be the Euler datum treated there for a finite extension F of $\boldsymbol{Q}$. Then, Theorem 1 of Part I states that $E(\bar{F}/F)$ is complete. The proof of this fact is divided into the following three steps (a)–(c):

(a) $E(\bar{F}/F)$ is complete iff $E(K/F)=(P(O_F), W(K/F), \alpha^K)$ are complete for all finite Galois extensions K of F;

(b) $E(K/F)=\overline{E(K/F)_1}$ with $E(K/F)_1=(P(O_F), W(K/F)_1, \alpha_1^K)$ where $W(K/F)_1$ denotes the compact subgroup of $W(K/F)$ consisting of elements of volume 1;

(c) $E(K/F)_1$ satisfies the condition L, so Theorem 1 is applicable.

The fact that $E(K/F)_1$ satisfies (I) and (II) is due to Weil. When we check (III), we obtain a good function $C(\rho)$ having an explicit expression using the conductor and the "archimedean parameters" of ρ.

The second example is an Euler datum of Selberg type. Let R be a compact Riemann surface of general type. Let $P=P(R)$ be the set of closed geodesics on R and define $N(p)=\exp(l(p))$ for $p \in P$ where $l(p)$ denotes the length of p. Let $E(R)=(P(R), \pi_1(R), \alpha)$ be the Euler datum where $\alpha(p) \in \mathrm{Conj}(\pi_1(R))$ is the conjugacy class of the fundamental group $\pi_1(R)$ determined by the loop p; we note that $0 \leqq \mu(P) \leqq 3/4 < d(P)=1$. Then, $E(R)$ satisfies the condition L. In fact, (I) and (II) are contained in Selberg's results (except for a slight modification of Euler products), and in (III) we take $C(\rho)=C_1 \deg(\rho)$ with a sufficiently large constant C_1. Hence we have:

Theorem 2. *$E(R)$ and $\overline{E(R)}$ are complete.*

Reference

[1] N. Kurokawa: On some Euler products. I. Proc. Japan Acad., **60**A, 335–338 (1984).