93. On Certain Cubic Fields. VI

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1. The notations E_F, E_F^+, $\mathcal{O}_F$ for an algebraic number field F, D_h for a polynomial $h(x)\in \boldsymbol{Z}[x]$ and $D_F(\alpha)$ for an algebraic number α in F have the same meanings as in [3].

In this note, we shall consider totally real cubic fields K with the properties:

(I) $\theta,\ \theta+1\in E_K$

(II) $\mathcal{O}_K=\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\theta^2$.

These fields will be called for convenience *primitive with two consecutive units*, in short *P-C* fields. We shall prove

Theorem. In *P-C* fields, we have $E_K=\langle\pm 1\rangle\times\langle\theta,\ \theta+1\rangle$.

2. Now we can distinguish four cases:

(1) $\theta,\ -1-\theta\in E_K^+$ (2) $\theta,\ 1+\theta\in E_K^+$

(3) $-\theta,\ -1-\theta\in E_K^+$ (4) $-\theta,\ 1+\theta\in E_K^+$

In the case (1), we have $N_{K/\boldsymbol{Q}}\theta=1$, $N_{K/\boldsymbol{Q}}(1+\theta)=-1$ which implies $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=x^3-mx^2-(m+3)x-1$, $m\in \boldsymbol{Z}$, and in the case (2), we have $N_{K/\boldsymbol{Q}}\theta=1$, $N_{K/\boldsymbol{Q}}(1+\theta)=1$ which implies $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=x^3-nx^2-(n+1)x-1$, $n\in\boldsymbol{Z}$. The cases (3), (4) can be reduced to the case (2) by replacing θ respectively by $-1-\theta$ and $-(1+\theta)^{-1}$. Accordingly, we have to consider two kinds of fields (*P-C*1) and (*P-C*2), which are *P-C* fields with properties (1) respectively (2).

Now we have

Theorem 1. *Cubic field* $K=\boldsymbol{Q}(\theta)$ *with* $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=f(x)\in\boldsymbol{Z}[x]$ *is* (*P-C*1) *field, if and only if* $f(x)=x^3-mx^2-(m+3)x-1$, $m\in\boldsymbol{Z}$ *and* $\sqrt{D_f}=m^2+3m+9$ *is square free.*

In fact, (1) is equivalent with $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=f(x)=x^3-mx^2-(m+3)x-1$ and in this case K is Galois and so totally real, and (II) holds if and only if $\sqrt{D_f}$ is square free.

Theorem 2. *Cubic field* $K=\boldsymbol{Q}(\theta)$ *with* $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=g(x)\in\boldsymbol{Z}[x]$ *is* (*P-C*2) *field, if and only if* $g(x)=x^3-nx^2-(n+1)x-1$, $n\in\boldsymbol{Z}$, $D_g=(n^2+n-3)^2-32>0$ *is square free.*

In fact, (2) is equivalent with $\mathrm{Irr}\,(\theta\,;\boldsymbol{Q})=x^3-nx^2-(n+1)x-1$ and $D_g>0$ means that K is totally real, and (II) means that D_g is square free.

3. *Proof of Theorem.* We shall prove this theorem in two cases: (*P-C*1) fields and (*P-C*2) fields.

(i) Case (P-$C1$). In [3], we have proved $E_K=\langle\pm 1\rangle\times\langle\theta,\ \theta+1\rangle$ for (P-$C1$) fields with $m\geqq -1$. The case $m<-1$ is reduced to this case for the following reason. Put $J(m,\ x)=x^3-mx^2-(m+3)x-1$ and $m+3=-l$. Then we have $-(1/x^3)J(m,\ x)=J(l,\ 1/x)$ and if $m\geqq -1$, then we have $l\leqq -2$. Thus if $\mathrm{Irr}\,(\theta\,;\ \boldsymbol{Q})=J(m,\ x)$ with $m\geqq -1$, then $\mathrm{Irr}\,(1/\theta,\ \boldsymbol{Q})=J(l,\ x)$ with $m<-1$.

(ii) Case (P-$C2$). In [4], we have proved $E_K=\langle\pm 1\rangle\times\langle\theta,\ \theta+1\rangle$ for (P-$C2$) fields with $n\leqq -7$. So we have to supplement the case $n=-5,\ -6$. The case $n\geqq 4$ is reduced to this case (see Remark 1 in [4]). Let S be the set of conjugate mappings of $K/\boldsymbol{Q}$. Using the fact $|z^n-1|\geqq\max(|z|,\ 1)^{n-2}||z|^2-1|$ for any $z\in\boldsymbol{C}$ and $n\in\boldsymbol{N}$ with $n\geqq 2$ (cf. [1]), we have $|\delta(\delta+1)-1|=|\lambda^3-1|\geqq\max(|\lambda|,\ 1)||\lambda|^2-1|$ in the notations of [4]. As $K/\boldsymbol{Q}$ is totally real, we have $|\lambda^\sigma|^2=(|\lambda|^\sigma)^2$ for any $\sigma\in S$, so that we have

$$(*)\qquad n^2+5n+5=\prod_{\sigma\in S}|(\delta(\delta+1)-1)^\sigma|=\prod_{\sigma\in S}\max(|\lambda|^\sigma,\ 1)\prod_{\sigma\in S}||\lambda^\sigma|^2-1|$$
$$>(|n+2||n+3|)^{1/3}|N_{K/\boldsymbol{Q}}(|\lambda|^2-1)|,$$

as the roots of $g(x)$ are situated as follows:

$$n+1<\delta_1<n+2,\quad -2<\delta_2<-1\quad\text{and}\quad 0<\delta_3<1.$$

A straightforward computation shows (see the proof of Theorem in [5] and consider the discriminants of $\mathrm{Irr}\,(|\lambda|\,;\ \boldsymbol{Q})$, $\mathrm{Irr}\,(|\lambda|-1\,;\ \boldsymbol{Q})$ and $\mathrm{Irr}\,(|\lambda|+1\,;\ \boldsymbol{Q})$), that we have $|N_{K/\boldsymbol{Q}}(|\lambda|^2-1)|\geqq 5$. Thus $(*)$ is impossible for $n=-5,\ -6$. Hence the case $(k,\ l)=(1,\ 1)$ can not take place.

Remark. Our theorem follows also from the following result of E. Thomas [2] (instead of [3], [4]): $K=\boldsymbol{Q}(\theta)$ with $\mathrm{Irr}\,(\theta\,;\ \boldsymbol{Q})=x^3-mx^2-(m+3)x-1$ with $m\geqq -1$ or $\mathrm{Irr}\,(\theta\,;\ \boldsymbol{Q})=x^3-(n-1)x^2+nx-1$ with $n\geqq 7$ has the property that $\langle\pm 1\rangle\times\langle\theta,\ \theta+1\rangle$ respectively $\langle\pm 1\rangle\times\langle\theta,\ \theta-1\rangle$ coincide with the unit groups of orders $\boldsymbol{Z}+\boldsymbol{Z}\theta+\boldsymbol{Z}\theta^2$. The proof is quite different from ours.

References

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