87. *The Global Hypoellipticity of a class of Degenerate Elliptic-Parabolic Operators*

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There are vast references on *local* hypoellipticity of degenerate elliptic-parabolic operators (cf. Amano [1], [2], Fedii [3], Hörmander [5], Morimoto [8], Oleinik and Radkevich [9] and their references), however one can find only few papers concerned with *global* hypoellipticity. Oleinik and Radkevich [9] and Kusuoka and Stroock [7] gave several criteria of global hypoellipticity. Fujiwara and Omori [4] found an operator which is globally hypoelliptic but not locally hypoelliptic. In this note, we give a criterion of global hypoellipticity which is finer than Oleinik and Radkevich's result, and show a theorem as one of its applications. Fujiwara and Omori's result is contained in our theorem as a special case. Oleinik and Radkevich's and Kusuoka and Stroock's theorems are not applicable to the operators treated in our theorem. We can apply our criterion to the wider class of degenerate elliptic-parabolic operators.

Let P be a differential operator of the form

$$P=\partial_1^2+a(x)\partial_2^2$$

with nonnegative coefficient $a(x)$ in $C^\infty(\boldsymbol{T}^2)$, where $\partial_i=\partial/\partial x_i$ $(i=1, 2)$ and $\boldsymbol{T}^2$ is the 2-dimensional torus $\boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$. X_0, X_1, X_2 denote vector fields defined by

$$X_0=-(\partial_2 a(x))\partial_2, \quad X_1=\partial_1, \quad X_2=a(x)\partial_2$$

and S is a subset of $\boldsymbol{T}^2$ defined by

$$S=\{x\in\boldsymbol{T}^2 : \dim \mathrm{Lie}\,(x)<2\},$$

where $\mathrm{Lie}\,(x)=\{X(x) : X\in \mathrm{Lie}\,(X_0, X_1, X_2)\}$.

Theorem. *Assume that*

$$(1) \qquad \partial_2^2 a(x)=0 \qquad \text{on } S.$$

Then the operator

$$P=\partial_1^2+a(x)\partial_2^2$$

is globally hypoelliptic in $\boldsymbol{T}^2$ if and only if the system

$$(2) \qquad \dot{x}=\sum_{i=0}^{2}\xi_i X_i(x), \qquad \xi_i\in\boldsymbol{R}$$

is controllable in $\boldsymbol{T}^2$.

Remark. The author does not know whether or not Theorem is valid without the assumption (1). It is to be noted that Theorem remains valid in case the set S and its boundary ∂S are nonsmooth.

If $S=\phi$, then the operator P is locally hypoelliptic in $\boldsymbol{T}^2$ (cf. Oleinik and Radkevich [9]), and further, if the operator P is locally hypoelliptic in $\boldsymbol{T}^2$, then the system (2) is controllable in every subdomain of $\boldsymbol{T}^2$ (cf. Amano [1]).

1. A criterion of global hypoellipticity. In this section, let Ω be an open set of $\boldsymbol{R}^d$ and let P be a degenerate elliptic-parabolic operator of the form

$$P=\sum_{i,j=1}^{d} a^{ij}(x)\partial_i\partial_j+\sum_{i=1}^{d} b^i(x)\partial_i+c(x)$$

with real coefficients in $C^\infty(\Omega)$. $X_0, X_1, \cdots, X_d$ denote vector fields defined by

$$X_0=\sum_{i=1}^{d}\Big(b^i-\sum_{j=1}^{d}\partial_j a^{ij}\Big)\partial_i,$$

$$X_1=\sum_{i=1}^{d} a^{1i}\partial_i,$$

$$\cdots$$

$$X_d=\sum_{i=1}^{d} a^{di}\partial_i.$$

Lie (x) is a distribution (in the sense of differential geometry) defined by

$$\text{Lie}\,(x)=\{X(x): X\in \text{Lie}\,(X_0, X_1, \cdots, X_d)\},$$

where Lie $(X_0, X_1, \cdots, X_d)$ is the Lie algebra generated by the vector fields $X_0, X_1, \cdots, X_d$. S denotes a set of all points x of Ω such that $\dim \text{Lie}\,(x)<d$. $\|\cdot\|_s$ $(s\in\boldsymbol{R})$ stands for the Sobolev norm. $P_{(\beta)}$ is a differential operator defined by $P_{(\beta)}=[\partial_x^\beta, P]$.

Proposition. *Assume that S is compact, and assume that for any $\delta>0$, for any multi-index β $(1\leq|\beta|\leq 2)$ and for any $N>0$ there is an open neighborhood U of S in Ω such that*

(3) $$\|u\|_0\leq C(\|Pu\|_0+\|u\|_{-N}), \qquad u\in C_0^\infty(U)$$

and

(4) $$\|P_{(\beta)}u\|_{-|\beta|}\leq\delta\|Pu\|_0+C\|u\|_{-N}, \qquad u\in C_0^\infty(U),$$

where C is a nonnegative constant depend on δ, β, N and U. Then the operator P is globally hypoelliptic in Ω.

This proposition is a re-formation of theorems given by Fedii [3], Morimoto [8] and Oleinik and Radkevich [9].

2. An example of globally hypoelliptic operator. Let P be a differential operator of the form

$$P=\partial_1^2+a(x)\partial_2^2$$

with nonnegative coefficient $a(x)$ in $C^\infty(\boldsymbol{T}^2)$.

Lemma 1. *For any compact set K of $\boldsymbol{R}^2$*

$$\|u\|_0^2\leq\frac{1}{4}(\text{diam}\,K)^2(2\|Pu\|_0^2+\sup_K|\partial_2^2 a|\,\|u\|_0^2), \qquad u\in C_K^\infty(\boldsymbol{R}^2),$$

Lemma 1 follows immediately from Poincaré's inequality.

Lemma 2. *For any compact set K of* $\boldsymbol{R}^2$

$$\sum_{|\beta|=1} \|P_{(\beta)}u\|_0^2 \leq 2\left(\sum_{i=1}^{2} \sup_K |\partial_i^2 a|\right) \|Pu\|_1 \|u\|_1, \qquad u \in C_K^\infty(\boldsymbol{R}^2).$$

Since

$$|\partial_i a(x)|^2 \leq 2(\sup_K |\partial_i^2 a|) a(x) \qquad (x \in K,\ i=1, 2),$$

we easily have Lemma 2.

Proof of Theorem. Lemmas 1, 2 and (1) imply (3) and (4). Hence, the "if" part of Theorem follows from Proposition. It is easy to construct a solution $u \notin C^\infty(\boldsymbol{T}^2)$ satisfying $Pu \in C^\infty(\boldsymbol{T}^2)$, when the system (2) is not controllable in $\boldsymbol{T}^2$; the "only if" part of Theorem is proved.

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