70. General Solutions of Witten's Gauge-field Equations

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§0. Introduction. Consider a gauge field in the eight-dimensional space satisfying

$$(1)\quad \begin{aligned}&[-\lambda_1\nabla_{\eta_1}+\nabla_{\zeta_1},\ \lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1}]=0, \qquad [-\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\end{aligned}$$

for any $\lambda_1, \lambda_2 \in \boldsymbol{C}$. Here $(\eta_1, \bar\eta_1, \zeta_1, \bar\zeta_1, \eta_2, \bar\eta_2, \zeta_2, \bar\zeta_2) \in \boldsymbol{C}^8$ and ∇_{η_1} etc. are covariant derivatives. To the gauge fields satisfying (1) corresponds a class of solutions of the Yang-Mills equations including all the self-dual or anti-self-dual solutions, when they are restricted to the four-dimensional subspace $\eta_1-\eta_2=\bar\eta_1-\bar\eta_2=\zeta_1-\zeta_2=\bar\zeta_1-\bar\zeta_2=0$ (cf. [1], [2]).

In our previous paper [2], we regarded (1) as the integrability condition for some linear equations with a pair of spectral parameters λ_1, λ_2. But in this paper, we note that the equations (1) imply

$$(2)\quad \begin{aligned}&[-\lambda\nabla_{\eta_1}+\nabla_{\zeta_1},\ \lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1}]=0, \qquad [-\lambda\nabla_{\eta_2}+\nabla_{\zeta_2},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[-\lambda\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ -\lambda\nabla_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[\lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ -\lambda\nabla_{\eta_2}+\nabla_{\zeta_2}]=[\lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\end{aligned}$$

for any $\lambda \in \boldsymbol{C}$. (This system of equations is classified into the class A_4 according to R. S. Ward [3].)

Therefore, the solution space of (1) is embedded in the solution space of (2), and it is sufficient to solve the following problems:

(i) to construct the general solutions of (2) and clarify their structure,

(ii) to characterize the solutions of (1) in the solution space of (2).

The problem (i) can be solved more simply than to investigate directly the solution space of (1) which we did in our previous paper [2], because the equations (2) comprise only one spectral parameter. In fact, we can solve them by direct application of Sato-Takasaki method (cf. [4], [5], [6]): we rewrite (2) into a system of differential equations for unknown functions valued in an infinite-dimensional Grassmann manifold and investigate the structure of its solution space by considering an initial-value problem with respect to the subspace $\bar\zeta_1=\bar\eta_1=\zeta_2=\bar\eta_2=0$.

The problem (ii) also can be solved and a simple characterization can be obtained. (See Theorem 3.)

§ 1. General solutions of the gauge-field equations A_4. Throughout this paper we discuss in the category of formal power series, so that $\nabla_{\eta_1}=\partial_{\eta_1}+A_{\eta_1}$, $A_{\eta_1}\in \mathrm{gl}(n, C[[\eta_1, \bar{\eta}_1, \cdots, \bar{\zeta}_2]])$ etc. The gauge-field equations (2) imply $[\nabla_u, \nabla_v]=0$ for $u, v=\eta_1, \zeta_1, \eta_2, \zeta_2$. Therefore we can "fix" the gauge, namely, restrict the freedom of gauge, so that $A_{\eta_1}=A_{\zeta_1}=A_{\eta_2}=A_{\zeta_2}=0$. Then (2) reads

$$(3)\quad \begin{aligned}&[-\lambda\partial_{\eta_1}+\nabla_{\bar{\zeta}_1}, \lambda\partial_{\zeta_1}+\nabla_{\bar{\eta}_1}]=0, \quad [-\lambda\partial_{\eta_2}+\nabla_{\bar{\zeta}_2}, \lambda\partial_{\zeta_2}+\nabla_{\bar{\eta}_2}]=0,\\ &[-\lambda\partial_{\eta_1}+\nabla_{\bar{\zeta}_1}, -\lambda\partial_{\eta_2}+\nabla_{\bar{\zeta}_2}]=[-\lambda\partial_{\eta_1}+\nabla_{\bar{\zeta}_1}, \lambda\partial_{\zeta_2}+\nabla_{\bar{\eta}_2}]=0,\\ &[\lambda\partial_{\zeta_1}+\nabla_{\bar{\eta}_1}, -\lambda\partial_{\eta_2}+\nabla_{\bar{\zeta}_2}]=[\lambda\partial_{\zeta_1}+\nabla_{\bar{\eta}_1}, \lambda\partial_{\zeta_2}+\nabla_{\bar{\eta}_2}]=0 \quad \text{for any } \lambda\in C.\end{aligned}$$

That is nothing but the integrability condition for the linear equations

$$(4)\quad \begin{aligned}&(-\lambda\partial_{\eta_1}+\nabla_{\bar{\zeta}_1})w(\lambda)=0, \quad (\lambda\partial_{\zeta_1}+\nabla_{\bar{\eta}_1})w(\lambda)=0,\\ &(-\lambda\partial_{\eta_2}+\nabla_{\bar{\zeta}_2})w(\lambda)=0, \quad (\lambda\partial_{\zeta_2}+\nabla_{\bar{\eta}_2})w(\lambda)=0.\end{aligned}$$

Proposition 1. *$A_{\bar{\zeta}_1}, A_{\bar{\eta}_1}, A_{\bar{\zeta}_2}, A_{\bar{\eta}_2}\in \mathrm{gl}(n, C[[\eta_1, \cdots, \bar{\zeta}_2]])$ are solutions of* (3) *if and only if there exists a solution $w(\lambda)=\sum_{j\geq 0} w_j\lambda^{-j}$ of* (4) *such that $w_0=1$, namely, if and only if there exist $w_j\in \mathrm{gl}(n, C[[\eta_1, \cdots, \bar{\zeta}_2]])$ which satisfy $w_0=1$, $w_j=0$ if $j<0$, and*

$$(5)\quad \begin{aligned}&-\partial_{\eta_1}w_{j+1}+(\partial_{\bar{\zeta}_1}+A_{\bar{\zeta}_1})w_j=0, \quad \partial_{\zeta_1}w_{j+1}+(\partial_{\bar{\eta}_1}+A_{\bar{\eta}_1})w_j=0,\\ &-\partial_{\eta_2}w_{j+1}+(\partial_{\bar{\zeta}_2}+A_{\bar{\zeta}_2})w_j=0, \quad \partial_{\zeta_2}w_{j+1}+(\partial_{\bar{\eta}_2}+A_{\bar{\eta}_2})w_j=0\\ &\qquad\qquad \textit{for any } j\in \mathrm{Z}.\end{aligned}$$

When $j=0$, (5) reads

$$(6)\quad \begin{aligned}&-\partial_{\eta_1}w_1+A_{\bar{\zeta}_1}=0, \quad \partial_{\zeta_1}w_1+A_{\bar{\eta}_1}=0,\\ &-\partial_{\eta_2}w_1+A_{\bar{\zeta}_2}=0, \quad \partial_{\zeta_2}w_1+A_{\bar{\eta}_2}=0.\end{aligned}$$

Therefore, to solve the eqs. (3), it is sufficient to solve the equations

$$(7)\quad \begin{aligned}&-\partial_{\eta_1}w_{j+1}+\partial_{\bar{\zeta}_1}w_j+(\partial_{\eta_1}w_1)w_j=0, \quad \partial_{\zeta_1}w_{j+1}+\partial_{\bar{\eta}_1}w_j-(\partial_{\zeta_1}w_1)w_j=0,\\ &-\partial_{\eta_2}w_{j+1}+\partial_{\bar{\zeta}_2}w_j+(\partial_{\eta_2}w_1)w_j=0, \quad \partial_{\zeta_2}w_{j+1}+\partial_{\bar{\eta}_2}w_j-(\partial_{\zeta_2}w_1)w_j=0\\ &\qquad\qquad \text{for any } j\in \mathrm{Z}.\end{aligned}$$

More precisely, we have

Proposition 2. *The relations* (6) *give a one-to-one correspondence between*

(i) *solutions $A=(A_{\bar{\zeta}_1}, A_{\bar{\eta}_1}, A_{\bar{\zeta}_2}, A_{\bar{\eta}_2})$ of* (3),

and

(ii) *equivalence classes of the solutions $w(\lambda)=1+\sum_{j>0} w_j\lambda^{-j}$ of* (7) *modulo right-multiplication by $v(\lambda)=1+\sum_{j>0} v_j\lambda^{-j}$ such that*

$$(-\lambda\partial_{\eta_1}+\partial_{\bar{\zeta}_1})v(\lambda)=(\lambda\partial_{\zeta_1}+\partial_{\bar{\eta}_1})v(\lambda)=(-\lambda\partial_{\eta_2}+\partial_{\bar{\zeta}_2})v(\lambda)$$
$$=(\lambda\partial_{\zeta_2}+\partial_{\bar{\eta}_2})v(\lambda)=0.$$

For any $w(\lambda)\in \mathrm{gl}(n, C[[\eta_1, \cdots, \bar{\zeta}_2]])[[\lambda^{-1}]]$ such that $w_0=1$, we define a matrix of infinite size $\xi=(\xi_{ij})_{i\in\mathrm{Z}, j<0}$ by the product of matrices $(w^*_{i-j})_{i\in\mathrm{Z}, j<0}$ and $(w_{i-j})_{i<0, j<0}$, i.e. by $\xi_{ij}=\sum_{k<0} w^*_{i-k}w_{k-j}$ where w^*_j are coefficients of $w(\lambda)^{-1}=\sum_{j\geq 0} w^*_j\lambda^{-j}$.

Then we have

Proposition 3. *The above definition of ξ gives a one-to-one correspondence between*

(i) $w(\lambda) \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \cdots, \bar{\zeta}_2]])[[\lambda^{-1}]]$ *such that* $w_0 = 1$
and

(ii) $\xi = (\xi_{ij})_{i \in Z, j<0}$ *such that* $\xi_{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \cdots, \bar{\zeta}_2]])$,

(8) $\xi_{ij} = \delta_{ij}$ *if* $i<0$, *and* $\Lambda\xi = \xi C$ *for some* $N^c \times N^c$-matrix $C = (C_{ij})_{i<0, j<0}$.

Here δ_{ij} *denotes the Kronecker's delta,* $\Lambda = (\delta_{i+1,j})_{i,j \in Z}$, *and* N^c *denotes the set of negative integers.*

Theorem 1. *Through the correspondence* $w \leftrightarrow \xi$, (7) *is equivalent to the existence of* $N^c \times N^c$-matrices A_1, A_2, B_1, B_2 *such that*

(9)
$$(-\Lambda\partial_{\eta_1} + \partial_{\zeta_1})\xi = \xi A_1, \qquad (\Lambda\partial_{\zeta_1} + \partial_{\bar{\eta}_1})\xi = \xi B_1,$$
$$(-\Lambda\partial_{\eta_2} + \partial_{\zeta_2})\xi = \xi A_2, \qquad (\Lambda\partial_{\bar{\zeta}_2} + \partial_{\bar{\eta}_2})\xi = \xi B_2,$$

A_1, A_2, B_1, B_2 *are uniquely determined by* ξ *if they exist.*

To investigate the structure of the solution space of (7) (or (9)), we consider an initial-value problem with respect to the subspace $\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0$.

Theorem 2. *For any datum* $\xi^{(0)} = (\xi_{ij}^{(0)})_{i \in Z, j<0}$, $\xi_{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1, \eta_2, \zeta_2]])$ *satisfying* (8), *there exists a unique solution* ξ *to the initial-value problem,* i.e. $\xi = (\xi_{ij})_{i \in Z, j<0}$, $\xi_{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \bar{\eta}_1, \cdots, \bar{\zeta}_2]])$ *satisfying* (8), (9), *and* $\xi|_{\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0} = \xi^{(0)}$. *The solution* ξ *can be obtained by the following formulae: let* $\tilde{\xi} = \exp[\Lambda(\bar{\zeta}_1\partial_{\eta_1} - \bar{\eta}_1\partial_{\zeta_1} + \zeta_2\partial_{\eta_2} - \bar{\eta}_2\partial_{\bar{\zeta}_2})]\xi^{(0)}$ *and* $\tilde{\xi}^{(-)} = (\tilde{\xi}_{ij})_{i<0, j<0}$, *then* $\xi = \tilde{\xi}(\tilde{\xi}^{(-)})^{-1}$.

Corollary. *For any datum* $w^{(0)}(\lambda) \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1, \eta_2, \bar{\zeta}_2]])[[\lambda^{-1}]]$ *such that* $w_0^{(0)} = 1$, *there exists a unique solution* $w(\lambda)$ *to the initial-value problem,* i.e. $w(\lambda) = 1 + \sum_{j>0} w_j \lambda^{-j} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \bar{\eta}_1, \cdots, \bar{\zeta}_2]])[[\lambda^{-1}]]$ *satisfying* (7) *and* $w(\lambda)|_{\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0} = w^{(0)}(\lambda)$.

§2. Characterization of Witten's gauge fields. A gauge field ∇ satisfies (1) if and only if it satisfies (2) and

(10)
$$[\nabla_{\eta_1}, \nabla_{\zeta_2}] = [\nabla_{\eta_1}, \nabla_{\bar{\eta}_2}] = [\nabla_{\zeta_1}, \nabla_{\zeta_2}] = [\nabla_{\zeta_1}, \nabla_{\bar{\eta}_2}] = 0.$$

Taking (6) into consideration, we obtain

Proposition 4. *To any solution* $w(\lambda)$ *of* (7) *corresponds a solution of* (1) *if and only if* $w_1 = \phi(\eta_1, \bar{\eta}_1, \zeta_1, \bar{\zeta}_1, \bar{\eta}_2, \zeta_2) + \psi(\bar{\eta}_1, \bar{\zeta}_1, \eta_2, \bar{\eta}_2, \zeta_2, \bar{\zeta}_2)$ *for some* $\phi \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \bar{\eta}_1, \zeta_1, \bar{\zeta}_1, \bar{\eta}_2, \zeta_2]])$, $\psi \in \mathrm{gl}(n, \boldsymbol{C}[[\bar{\eta}_1, \bar{\zeta}_1, \eta_2, \bar{\eta}_2, \zeta_2, \bar{\zeta}_2]])$.

The corollary of Theorem 2 claims that the solution space of (7) is parametrized by the totality of arbitrary initial data $w^{(0)}(\lambda) \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1, \eta_2, \bar{\zeta}_2]])[[\lambda^{-1}]]$ such that $w_0^{(0)} = 1$. Thus we can clarify the structure of the solution space by considering the initial data:

Theorem 3. (i) *To any initial datum* $w^{(0)}(\lambda)$ *corresponds a solution of* (1) *if and only if* $w_1^{(0)} = \phi(\eta_1, \zeta_1) + \psi(\eta_2, \bar{\zeta}_2)$ *for some* $\phi \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1]])$, $\psi \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_2, \bar{\zeta}_2]])$.

(ii) *To any datum* $w^{(0)}(\lambda)$ *corresponds a self-dual Yang-Mills field if and only if* $w_1^{(0)} = \phi(\eta_1, \zeta_1) + f(\eta_2) + g(\bar{\zeta}_2)$ *for some* $\phi \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1]])$, $f(X), g(X) \in \mathrm{gl}(n, \boldsymbol{C}[[X]])$.

(iii) *To any datum* $w^{(0)}(\lambda)$ *corresponds an anti-self-dual Yang-Mills field if and only if* $w_1^{(0)}=f(\eta_1)+g(\zeta_1)+\psi(\eta_2,\bar{\zeta}_2)$ *for some* $f(X), g(X) \in \mathrm{gl}\,(n, \boldsymbol{C}[[X]])$, $\psi(\eta_2,\bar{\zeta}_2)\in \mathrm{gl}\,(n, \boldsymbol{C}[[\eta_2,\bar{\zeta}_2]])$.

(iv) *To any datum* $w^{(0)}(\lambda)$ *corresponds a trivial solution* (i.e. *a flat connection*) *if and only if* $w_1^{(0)}=f(\eta_1)+g(\zeta_1)+h(\eta_2)+k(\bar{\zeta}_2)$ *for some* $f(X)$, $g(X), h(X), k(X)\in \mathrm{gl}\,(n, \boldsymbol{C}[[X]])$.

Remark. Starting from arbitrary rational functions $\phi(\eta_1,\zeta_1)$, $\psi(\eta_2,\bar{\zeta}_2)$ that cannot be decomposed like (ii) or (iii), we can construct a rational solution corresponding to a Yang-Mills field which is not self-dual nor anti-self-dual by the formulae in Theorem 2.

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