68. On the Banach-Saks Property

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1. Introduction. According to Banach and Saks [1], every bounded sequence in $L^p(0,1)$ or l^p $(1<p<\infty)$ has a subsequence whose Cesàro-means converge strongly. More generally every uniformly convex Banach space possesses this so-called *Banach-Saks property*, as shown by Kakutani [4]. In particular every Hilbert space has this property. In nonlinear analysis, by utilizing a duality mapping some assertions which are valid in the case of Hilbert spaces are extended to the case of special classes of Banach spaces. Especially in the case of Banach spaces with a uniformly convex conjugate space, such extentions are often obtained since a duality mapping is uniformly strongly continuous on each bounded subset of such a Banach space (see Browder [2, p. 42] or Kato [5]). So we consider whether such a Banach space has the Banach-Saks property or not. The result is positively extended and is stated as follows:

Theorem. *Let X be a Banach space with a uniformly convex conjugate space X^*. Then X possesses the Banach-Saks property.*

After we have proved the above theorem, we find the following result due to Enflo [3]:

For a Banach space X with a conjugate space X^*, X is uniformly convexifiable if and only if X^* is uniformly convexifiable.

Hence, combining this result and Kakutani's theorem we can get our theorem. However, our method of the proof is based on very elementary facts about a duality mapping and there might be some interest in the simplicity of the construction of a subsequence whose Cesàro-means converge strongly.

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2. Proof of the theorem.

Proof of the theorem. Since X^* is uniformly convex, for each x in X there exists a unique $F(x)$ in X^* such that

$$(x, F(x)) = \|x\|\,\|F(x)\| \quad \text{and} \quad \|x\| = \|F(x)\|,$$

where $(\cdot,\cdot)$ denotes the canonical pairing of X and X^*. The mapping $F: X \to X^*$ is the so-called duality mapping. As we stated in the introduction, the uniform convexity of X^* also implies that F is uniformly strongly continuous on each bounded subset of X. Since X

is reflexive, all we have to show is the following:

If a sequence $\{x_n\}_n$ $(n=1, 2, \cdots)$ *in* X *converges weakly to zero, then* $\{x_n\}_n$ *has a subsequence whose Cesàro-means converge strongly to zero.*

For such a sequence $\{x_n\}_n$, we put

$$r := \sup_{n \in N} \|x_n\| \quad \text{and} \quad B := \{x \in X : \|x\| \leq r\}.$$

Since $\{x_n\}_n$ converges weakly to zero, we can choose a subsequence, which we still denote by $\{x_n\}_n$ such that

$$\operatorname{Re}(x_n, F(x_1+x_2+\cdots+x_{n-1}))<1 \qquad \text{for all } n\geq 2.$$

We shall show that this subsequence meets the requirement. To this end we set

$$S_n := \sum_{j=1}^{n} x_j \qquad (n\geq 1).$$

Since F is uniformly strongly continuous on B, for any $\varepsilon>0$ there exists a $\delta>0$ such that for any x, y in B with $\|x-y\|<\delta$,

$$\|F(x)-F(y)\|<\varepsilon/r.$$

Take an integer $m>r/\delta$, then it follows that for all $n>m$,

$$\begin{aligned}&\operatorname{Re}(S_n-S_{n-1}, F(S_n/n)-F(S_{n-1}/n))\\ &\leq \|S_n-S_{n-1}\| \, \|F(S_n/n)-F(S_{n-1}/n)\|\\ &< r\cdot \varepsilon/r=\varepsilon.\end{aligned}$$

Hence we have

$$\operatorname{Re}(S_n-S_{n-1}, F(S_n)-F(S_{n-1}))<\varepsilon n \qquad (n>m).$$

On the other hand,

$$\begin{aligned}&\operatorname{Re}(S_n-S_{n-1}, F(S_n)-F(S_{n-1}))\\ &=[\|S_n\|^2+\|S_{n-1}\|^2-2\|S_n\| \, \|S_{n-1}\|]\\ &\quad+[\|S_n\| \, \|F(S_{n-1})\|-\operatorname{Re}(S_n, F(S_{n-1}))]\\ &\quad+[\|S_{n-1}\| \, \|F(S_n)\|-\operatorname{Re}(S_{n-1}, F(S_n))],\end{aligned}$$

where each of the terms in square brackets is nonnegative. Hence we see that

$$\|S_n\| \, \|F(S_{n-1})\|-\operatorname{Re}(S_n, F(S_{n-1}))<\varepsilon n$$

or

$$\|S_n\| \, \|S_{n-1}\|-\|S_{n-1}\|^2-\operatorname{Re}(x_n, F(S_{n-1}))<\varepsilon n \qquad (n>m).$$

Therefore we get

$$\|S_n\| \, \|S_{n-1}\|-\|S_{n-1}\|^2<\varepsilon n+1,$$

hence

$$\begin{aligned}\|S_n\|^2-\|S_{n-1}\|^2&=(\|S_n\|-\|S_{n-1}\|)^2+2(\|S_n\| \, \|S_{n-1}\|-\|S_{n-1}\|^2)\\ &<r^2+2(\varepsilon n+1),\end{aligned}$$

and hence

$$\begin{aligned}\|S_n\|^2&<\|S_m\|^2+\sum_{j=m+1}^{n}(r^2+2+2\varepsilon j)\\ &<\|S_m\|^2+(r^2+2)n+\varepsilon n(n+1) \qquad (n>m).\end{aligned}$$

Thus we obtain

$$\|S_n\|^2/n^2<\|S_m\|^2/n^2+(r^2+2)/n+\varepsilon(n+1)/n \qquad \text{for all } n>m,$$

hence

$$\limsup_{n\to\infty} \|S_n\|^2/n^2\leq\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this means that

$$\lim_{n\to\infty} \|S_n\|/n=0.$$

This is the desired result and the proof of our theorem is complete.

References

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