60. Confluent Hypergeometric Functions on an Exceptional Domain

By Shōyū NAGAOKA
Department of Mathematics, Kinki University

(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1984)

In [3], G. Shimura studied the generalized confluent hypergeometric functions on tube domains of several types. A motive of his study can be seen in the application to the Eisenstein series as developed in his recent paper [4]. In this paper, we shall describe analogous results in the case of tube domains constructed from Cayley's octonion (which includes the case of exceptional simple tube domain).

We denote by $\mathfrak{C}_{\boldsymbol{R}}$ the real Cayley algebra, and we fix the standard basis (e.g. cf. [2]). For each integer m $(1\leqq m\leqq 3)$, we put $\kappa(m)=4m-3$. We define a vector space $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ over $\boldsymbol{R}$ by $\mathfrak{J}_{\boldsymbol{R}}^{(m)}=\{x\in \boldsymbol{M}_m(\mathfrak{C}_{\boldsymbol{R}})\,|\,{}^t\bar{x}=x\}$, where the bar denotes the Cayley conjugation. We supply $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ with a product by $x\circ y=(1/2)(xy+yx)$, with this product, $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ becomes a real Jordan algebra. When $m=3$, $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ is called the exceptional Jordan algebra (cf. [1]). If $x=(x_{ij})\in\mathfrak{J}_{\boldsymbol{R}}^{(m)}$, we define $\operatorname{tr}(x)=\sum x_{ii}\in\boldsymbol{R}$ and define an inner product $(\ ,\ )$ on $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ by $(x,y)=\operatorname{tr}(x\circ y)$. Moreover, we define a polynomial function det on $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ as follows. When $m=3$,

$$\det(x)=\prod_{i=1}^{3}x_{ii}-x_{11}\boldsymbol{N}(x_{23})-x_{22}\boldsymbol{N}(x_{13})-x_{33}\boldsymbol{N}(x_{12})+\boldsymbol{T}((x_{12}x_{23})\bar{x}_{13}),$$

where $\boldsymbol{N}(a)=a\bar{a}=\bar{a}a$, $\boldsymbol{T}(a)=a+\bar{a}$ $(a\in\mathfrak{C}_{\boldsymbol{R}})$. In the case $m=2$, we define as $\det(x)=x_{11}x_{22}-\boldsymbol{N}(x_{12})$. We denote by $\mathfrak{R}_m$ the set of squares $x\circ x$ of elements of $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$, and by $\mathfrak{R}_m^+$, the interior of $\mathfrak{R}_m$; then $\mathfrak{R}_m^+$ is a convex open cone in $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$. $\mathfrak{R}_3^+$ is called the exceptional cone. Identifying $\boldsymbol{C}^{m\kappa(m)}$ with $\mathfrak{J}_{\boldsymbol{C}}^{(m)}=\mathfrak{J}_{\boldsymbol{R}}^{(m)}\otimes_{\boldsymbol{R}}\boldsymbol{C}$, we define a tube domain $\boldsymbol{H}_m$ by $\boldsymbol{H}_m=\{x+iy\,|\,x\in\mathfrak{J}_{\boldsymbol{R}}^{(m)},\ y\in\mathfrak{R}_m^+\}$. Then $\boldsymbol{H}_3$ is the exceptional tube domain of type E_7 (cf. [1]) and $\boldsymbol{H}_1$ is the complex upper-half plane. We define a Euclidean measure dx on $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ by viewing $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ as $\boldsymbol{R}^{m\kappa(m)}$. Now we define the generalized gamma function $\Gamma_m(s)$ associated with the cone $\mathfrak{R}_m^+$ by

$$\Gamma_m(s)=\int_{\mathfrak{R}_m^+}e^{-\operatorname{tr}(x)}\det(x)^{s-\kappa(m)}dx,$$

then the integral converges for $\operatorname{Re}(s)>\kappa(m)-1$ and satisfies the following identity:

$$\Gamma_m(s)=\pi^{2m(m-1)}\prod_{n=0}^{m-1}\Gamma(s-4n),$$

where $\Gamma(s)$ is the ordinary gamma function (e.g. cf. [1]). Put, for $g\in\mathfrak{R}_m^+$, $h\in\mathfrak{J}_{\boldsymbol{R}}^{(m)}$, and $(\alpha,\beta)\in\boldsymbol{C}^2$,

$$\eta_m(g, h; \alpha, \beta)=\int_{x\pm h\in\mathfrak{R}_m^+} e^{-(g,x)} \det(x+h)^{\alpha-\kappa(m)} \det(x-h)^{\beta-\kappa(m)}dx,$$
$$\eta_m^*(g, h; \alpha, \beta)=\det(g)^{\alpha+\beta-\kappa(m)}\eta_m(g, h; \alpha, \beta).$$

We note that the function η_m represents by the generalized confluent hypergeometric function

$$\zeta_m(g; \alpha, \beta)=\int_{\mathfrak{R}_m^+} e^{-(g,x)} \det(\varepsilon_m+x)^{\alpha-\kappa(m)} \det(x)^{\beta-\kappa(m)}dx,$$

where $g\in\mathfrak{R}_m^+$ and ε_m is the identity matrix of degree m. We denote by $\mathfrak{J}_{\boldsymbol{R}}^{(m)}(p, q, r)$ the subset of $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ consisting of the elements with p positive, q negative, and r zero eigenvalues $(p+q+r=m)$. The precise definition of eigenvalue is as follows. When $m=3$, the eigenvalues of an element h of $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ are defined as the roots of a cubic equation $t^3-\mathrm{tr}(h)t^2+\mathrm{tr}(h\times h)t-\det(h)=0$, where $x\times y$ denotes the crossed product of $x, y\in\mathfrak{J}_{\boldsymbol{R}}^{(m)}$. In the case $m=2$, we define the eigenvalues of an element $h\in\mathfrak{J}_{\boldsymbol{R}}^{(2)}$ to be the roots of a quadratic equation $t^2-\mathrm{tr}(h)t+\det(h)=0$. Moreover, by similar way in [3], we shall introduce the notion of the eigenvalues of h *relative to* g for $h\in\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ and $g\in\mathfrak{R}_m^+$. In the case of degree 3, we define them to be the roots of an equation $t^3-(g, h)t^2+(g\times g, h\times h)t-\det(g)\det(h)=0$. When $m=2$, they are defined as the roots of an equation $t^2-(g, h)t+\det(g)\det(h)=0$. Now we denote by $\delta_+(hg)$ (resp. $\tau_+(hg)$) the product (resp. the sum) of all positive eigenvalues of h relative to g. Moreover, we put $\delta_-(hg)=\delta_+((-h)g)$, $\tau_-(hg)=\tau_+((-h)g)$ and $\tau(hg)=\tau_+(hg)+\tau_-(hg)$. We also denote by $\mu(hg)$ the smallest absolute value of non zero eigenvalues of h relative to g if $h=0$; $\mu(hg)=1$ if $h=0$. Now we define, for $g\in\mathfrak{R}_m^+$, $h\in\mathfrak{J}_{\boldsymbol{R}}^{(m)}(p, q, r)$, $(\alpha, \beta)\in\boldsymbol{C}^2$,

$$\begin{aligned}\omega_m(g, h; \alpha, \beta)=&2^{-p\alpha-q\beta}\Gamma_p(\beta-4(m-p))^{-1}\Gamma_q(\alpha-4(m-q))^{-1}\\ &\cdot\Gamma_r(\alpha+\beta-\kappa(m))^{-1}\delta_+(hg)^{\kappa(m)-\alpha-2q}\\ &\cdot\delta_-(hg)^{\kappa(m)-\beta-2p}\eta_m^*(g, h; \alpha, \beta),\end{aligned}$$

where we understand that Γ_0 is the constant function 1. The first main theorem can be stated as follows.

Theorem 1. *Function* ω_m *can be continued as a holomorphic function in* (α, β) *to the whole* $\boldsymbol{C}^2$ *and satisfies*

(1) $\omega_m(g, h; \alpha, \beta)=\omega_m(g, h; \kappa(m)+4r-\beta, \kappa(m)+4r-\alpha)$,

where r *is the number of zero eigenvalues of* h. *Moreover, for every compact set* $\boldsymbol{T}$ *of* $\boldsymbol{C}^2$, *there exist two positive constants* A *and* B *depending only on* $\boldsymbol{T}$ *such that*

(2) $|\omega_m(g, h; \alpha, \beta)|\leqq Ae^{-\tau(hg)/2}(1+\mu(hg)^{-B})$

for every $(g, h)\in\mathfrak{R}_m^+\times\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ *and every* $(\alpha, \beta)\in\boldsymbol{T}$.

This result is in analogy to Theorem 4.2 in [3].

Now consider a series

$$S_m(z, L_m; \alpha, \beta)=\textstyle\sum_{a\in L_m}\det(z+a)^{-\alpha}\det(\bar{z}+a)^{-\beta}.$$

Here z is a variable on $\boldsymbol{H}_m$, L_m is a lattice in the space $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ and (α, β)

$\in \boldsymbol{C}^2$. We see that this series is locally uniformly convergent on $\boldsymbol{H}_m \times \{(\alpha, \beta) \in \boldsymbol{C}^2 \mid \mathrm{Re}(\alpha+\beta) > 2\kappa(m)-1\}$. Following to [3], we introduce the notion of an algebraic lattice L in $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$, which means a lattice whose elements have algebraic components when we identify $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$ with $\boldsymbol{R}^{m\kappa(m)}$. Using Theorem 1, we can prove the following theorem

Theorem 2. *Let L be an algebraic lattice in $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$. Then*

$$\Gamma_m(\alpha+\beta-\kappa(m))^{-1} S_m(z, L\,;\,\alpha, \beta)$$

can be continued as a holomorphic function in (α, β) to the whole $\boldsymbol{C}^2$.

Now we put

$$S_m(z, L\,;\,\alpha) = \sum_{a \in L} \det(z+a)^{-\alpha},$$
$$S_m^*(z, L\,;\,\alpha) = \lim_{s \to 0} S_m(z, L\,;\,\alpha+s, s).$$

Then the series $S_m(z, L\,;\,\alpha)$ is convergent if $\mathrm{Re}(\alpha) > 2\kappa(m)-1$ and defines a holomorphic function in (z, α). Obviously, $S_m^*(z, L\,;\,\alpha)$ is equal to $S_m(z, L\,;\,\alpha)$ if $\mathrm{Re}(\alpha) > 2\kappa(m)-1$. Furthermore we have the following results.

Theorem 3. *Suppose L is an algebraic lattice in $\mathfrak{J}_{\boldsymbol{R}}^{(m)}$. Then $S_m^*(z, L\,;\,\alpha)$ coincides with $S_m(z, L\,;\,\alpha)$ for $\mathrm{Re}(\alpha) > \kappa(m)$. Moreover we have*

$$\mu(\mathfrak{J}_{\boldsymbol{R}}^{(m)}/L) S_m^*(z, L\,;\,\kappa(m)) = 2^{-4m(m-1)} i^{-m\kappa(m)} \Gamma_m(\kappa(m))^{-1} \sum 2^{-r(h)} e^{2\pi i (h, z)}$$

where the sum extends over all the elements in $L' \cap \mathfrak{J}_{\boldsymbol{R}}^{(m)}(p, 0, r)$ (L' is the dual lattice of L and $r(h)=r$) and $\mu(\mathfrak{J}_{\boldsymbol{R}}^{(m)}/L)$ is the measure of $\mathfrak{J}_{\boldsymbol{R}}^{(m)}/L$.

Finally we shall remark on an application of the above results. W. L. Baily, Jr. studied the Eisenstein series of the exceptional modular group Γ ([1]). Following his paper [1], we consider a series

$$\boldsymbol{E}(s, z) = \sum_{\gamma \in \Gamma/\Gamma_0} |j(z, \gamma)|^s, \qquad (s \in \boldsymbol{C}, z \in \boldsymbol{H}_3),$$

where $j(z, \gamma)$ is the functional determinant of the transformation γ at the point z and Γ_0 is a subset of Γ (cf. [1]). Now we put

$$\gamma_m(s) = \pi^{-ms} \Gamma_m(s) \quad \text{and} \quad \zeta_m(s) = \prod_{n=0}^{m-1} \zeta(s-4n),$$

where $\zeta(s)$ is the Riemann zeta function. Moreover we put

$$\xi(s) = \gamma_3(s/2) \zeta_3(s) \det(\mathrm{Im}(z))^{s/2} \boldsymbol{E}(s, z).$$

The Fourier coefficient of $\boldsymbol{E}(s, z)$ can be essentially expressed as a product of the "singular series" and the above-defined function ω_3. Therefore it is conjectured that the function $\xi(s)$ can be continued as a meromorphic function in s and satisfies a functional equation of the form

$$\xi(s) = \xi(18-s).$$

References

[1] W. L. Baily, Jr.: An exceptional arithmetic group and its Eisenstein series. Ann. of Math., **91**, 512–549 (1970).

[2] ——: Introductory Lectures on Automorphic Forms. Publication of Math. Soc. of Japan, **12** (1973).

[3] G. Shimura: Confluent hypergeometric functions on tube domains. Math. Ann., **260**, 269–302 (1982).

[4] ——: On Eisenstein series. Duke Math. J., **50**, 417–476 (1983).