58. Variational Problems Governed by a Multi-Valued Differential Equation

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1. Introduction. Throughout this paper, $\mathfrak{H}$ stands for a real Hilbert space of finite dimension and let a correspondence (=multi-valued mapping) $\Gamma:[0,T]\times\mathfrak{H}\twoheadrightarrow\mathfrak{H}$ be given. Define $\Delta(a)$ as a set of all elements x of $\mathcal{W}^{1,2}([0,T],\mathfrak{H})$ that satisfy

$$\begin{cases}\dot{x}(t)\in\Gamma(t,x(t)) & \text{a.e.}\\ x(0)=a\end{cases}\tag{$*$}$$

In the previous paper [1], we proved, under some assumptions, that $\Delta(a)$ is non-empty and it depends continuously, in some sense, upon the initial value a. In the present paper, we shall examine a couple of variational problems governed by a multi-valued differential equation of the form ($*$), and establish sufficient conditions which assure the existence of optimal solutions for them.

2. Review of the Previous Result. For the sake of the readers' convenience, we shall summarize the main result obtained in Maruyama [1].

Assumption 1. Γ is compact-convex-valued; i.e. $\Gamma(t,x)$ is a non-empty, compact and convex subset of $\mathfrak{H}$ for all $t\in[0,T]$ and all $x\in\mathfrak{H}$.

Assumption 2. The correspondence $x\twoheadrightarrow\Gamma(t,x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t\in\Gamma[0,T]$.

Assumption 3. The correspondence $t\twoheadrightarrow\Gamma(t,x)$ is measurable for each fixed $x\in\mathfrak{H}$.

For the concept of "measurability" of a correspondence, see Maruyama [6] Chap. 6.

Assumption 4. There exists $\psi\in L^2([0,T],\boldsymbol{R}_+)$ such that

$$\Gamma(t,x)\subset S_{\psi(t)}\qquad\text{for every }(t,x)\in[0,T]\times\mathfrak{H},$$

where $S_{\psi(t)}$ is the closed ball in $\mathfrak{H}$ with the center 0 and the radius $\psi(t)$.

Existence Theorem (Maruyama [1]). *Suppose that Γ satisfies Assumption 1-4, and let A be a non-empty, convex and compact subset of $\mathfrak{H}$. Then*

(i) *$\Delta(a^*)\neq\phi$ for any $a^*\in A$, and*

(ii) *the correspondence $\Delta:A\twoheadrightarrow\mathcal{W}^{1,2}$ is compact-valued and upper hemi-continuous (abbreviated as u.h.c.) on A in the weak topology for $\mathcal{W}^{1,2}$.*

3. Variational Problem (1). Let $u:[0,T]\times\mathfrak{H}\times\mathfrak{H}\to\boldsymbol{R}_+$ be a given function and consider the following variational problem:

(P-1) $$\operatorname*{Maximize}_{x \in \Delta(a)} J(x) = \int_0^T u(t, x(t), \dot{x}(t))dt$$

for a given $a \in A$.

We shall begin by specifying the properties of u.

Assumption 5. $-u$ is a normal integrand; i.e. a) $u(t, x, y)$ is measurable in (t, x, y) and b) the function $(x, y) \to u(t, x, y)$ is upper semi-continuous (abbreviated as u.s.c.) for every fixed $t \in [0, T]$.

Assumption 6. The function $(x, y) \to u(t, x, y)$ is concave for every fixed $t \in [0, T]$.

Assumption 7. There exists a non-negative function $\theta \in L^1([0, T], \boldsymbol{R})$ and a number b such that $u(t, x, y) - b \cdot \psi(t) \leqq \theta(t)$ for every $(t, x, y) \in [0, T] \times \mathfrak{H} \times \mathfrak{H}$.

Remark. 1) $\Delta(A) = \bigcup_{a \in A} \Delta(a)$ is a weakly compact set in $\mathcal{W}^{1,2}$ because it is an image of a compact set $A \subset \mathfrak{H}$ under the compact-valued u.h.c. correspondence Δ.

2) Under the Assumption 7, it is clear that

$$\sup_{x \in \Delta(a)} J(x) < \infty.$$

Lemma. *Under the Assumption* 1-7, *the integral functional J is u.s.c. on $\Delta(A)$ in the weak topology for $\Delta(A)$.*

Proof. First we must note that the bounded set $\Delta(A)$ endowed with the weak topology is metrizable since $\mathcal{W}^{1,2}$ is a separable Hilbert space (Maruyama [2] p. 357). Let $\{x_n\}$ be a sequence in $\Delta(A)$ which weakly converges to some $x^* \in \Delta(A)$. We have to show that

$$\limsup_n J(x_n) \leqq J(x^*).$$

Without loss or generality, we may assume that

$$\lim_n J(x_n) = \limsup_n J(x_n).$$

Since $x_n \to x^*$ weakly in $\mathcal{W}^{1,2}$, we may assume, again without loss of generality, that and

(1) $$x_n(t) \to x^*(t) \quad \text{a.e.}$$

(2) $$\dot{x}_n \to \dot{x}^* \qquad \text{weakly in } L^2.$$

Applying the Mazur's Theorem, we can find out, for each $j \in \boldsymbol{N}$, some finite elements

$$x_{n_j+1}, x_{n_j+2}, \cdots, x_{n_j+m(j)}$$

in $\{x_n\}$ and

$$\alpha_{i_j} \geqq 0,\ 1 \leqq i \leqq m(j),\ \textstyle\sum_{i=1}^{m(j)} \alpha_{i_j} = 1$$

such that

$$\begin{cases} \| \dot{x}^* - \sum_{i=1}^{m(j)} \alpha_{i_j} \dot{x}_{n_j+i} \| L^2 \leqq 1/j \\ n_{j+1} \geqq n_j + m(j). \end{cases}$$

Accordingly, if we denote

$$\xi_j(t) = \textstyle\sum_{i=1}^{m(j)} \alpha_{i_j} \dot{x}_{n_j+i}(t),$$

there is a subsequence (no change in notation) of $\{\xi_j\}$ such that

(3) $$\xi_j(t) \to \dot{x}^*(t) \quad \text{a.e.}$$

By the Assumption 6,

(4) $$\sum_{i=1}^{m(j)} \alpha_{ij} u(t, x_{n_j+i}(t), \dot{x}_{n_j+i}(t)) \leqq u(t, \sum_{i=1}^{m(j)} \alpha_{ij} x_{n_j+i}(t), \sum_{i=1}^{m(j)} \alpha_{ij} \dot{x}_{n_j+i}(t)).$$

For the sake of the simplicity of notations, we put

$$\zeta_j(t) \equiv \text{RHS} \qquad \text{of (4)}.$$

By (1), (3), and u.s.c. of u (Assumption 5), we have

(5) $$\limsup_j \zeta_j(t) \leqq u(t, x^*(t), \dot{x}^*(t)) \quad \text{a.e.}$$

Hence

$$\begin{aligned}\limsup_n J(x_n) &= \limsup_j \int_0^T \sum_{i=1}^{m(j)} \alpha_{ij} u(t, x_{n_j+i}(t), \dot{x}_{n_j+i}(t))dt \\ &\underset{(6)}{\leqq} \limsup_j \int_0^T \zeta_j(t)dt \underset{(7)}{\leqq} \int_0^T \limsup_j \zeta_j(t)dt \\ &\underset{(8)}{\leqq} \int_0^T u(t, x^*(t), \dot{x}^*(t))dt = J(x^*),\end{aligned}$$

where the inequality (6) comes from (4), (7) from the Fatou's lemma, and (8) from (5). This proves the lemma. Q.E.D.

Combining the Existence Theorem for the differential equation (*) together with the above Lemma, we obtain the following theorem.

Theorem 1. *Under the Assumption* 1-7, *the problem* (P-1) *has a solution.*

4. Variational Problem (2). Finally we consider a different kind of variational problem, in which the initial value is also variable:

(P-2) $$\underset{a \in A, x \in \Delta(a)}{\text{Maximize}} \int_0^T u(t, x(t), \dot{x}(t))dt.$$

In order to solve such a two-stage maximization problem as (P-2), the celebrated Berge's Maximum Theorem plays a crucial role.

Berge's Theorem. *Let Y and Z be any topological spaces. And assume that the function $f: Y \times Z \to \boldsymbol{R}$ is u.s.c. and the correspondence $\theta: Y \twoheadrightarrow Z$ is compact-valued and u.h.c. Then the function $f^*: Y \to R$ defined by*

$$f^*(y) = \text{Max}\{f(y, z) \mid z \in \theta(y)\}$$

is u.s.c. on Y. (c.f. Maruyama [2] Chap 2, §4.)

Applying Berge's theorem to our problem (P-2), we can assert that the function

$$a \to \underset{x \in \Delta(a)}{\text{Max}} \int_0^T u(t, x(t), \dot{x}(t))dt$$

is u.s.c. on A. Thus we get the following.

Theorem 2. *Under the Assumption* 1-7, *the problem* (P-2) *has a solution.*

References

[1] Maruyama, T.: On a multi-valued differential equation: an existence theorem. Proc. Japan Acad., **60A**, 161–164 (1984).

[2] ——: Functional Analysis. Keio Tsushin, Tokyo (1980) (in Japanese).