54. Huygens' Principle for a Generalized Euler-Poisson-Darboux Equation*)

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1. Introduction. The following pseudo-Riemannian metric is considered in $\boldsymbol{R}^n$;

(1.1) $$(ds)^2 = 2dx^1dx^2 - \sum_{3\leqq \alpha,\beta\leqq n} a_{\alpha\beta}(x^1)dx^\alpha dx^\beta,$$

where $(a_{\alpha\beta})_{3\leqq\alpha,\beta\leqq n}$ is a symmetric, positive definite and C^∞ matrix depending only on the first variable x^1. Let $L(x^3, \cdots, x^n)$ be a non-trivial linear form in $\boldsymbol{R}^n$, depending only on $(x^3, \cdots, x^n)$, and $M=\boldsymbol{R}^n\backslash\mathrm{Ker}\, L$ be a pseudo-Riemannian manifold with the Lorentzian metric (1.1). Let $\sigma(x^1)$ be a non-vanishing C^∞ function of x^1. On M we define a wave equation with the following lower order term;

(1.2) $$Pu = \Box_M u + 2\langle \nabla \log(\sigma(x^1)L(x)^\lambda), \nabla u\rangle, \qquad \lambda \in \boldsymbol{C}.$$

Here, $\Box_M$ is the ordinary wave equation on M determined by the metric (1.1) and ∇u denotes the gradient of u. The inner product $\langle , \rangle$ defined in the tangent space $T_p(M)$ induces an inner product, denoted by the same notation, in the cotangent space by the obvious isomorphism between both spaces. We shall write $|v|^2=\langle v, v\rangle$ for a tangent or cotangent vector v.

The metric (1.1) was employed by Günther [3] in order to obtain non-trivial examples of Huygens' principle. He proved that, in case $n=4$, $\Box_M u=0$ becomes a Huygens differential equation (written HDE for short) in the following sense. Let $E(x, y)$ be the forward fundamental solution of $\Box_M$; $\Box_M E=\delta(x-y)$, supp $E\subset$ the future propagation cone with vertex y, (the time orientation can be fixed in an arbitrarily way). Then $E(x, y)$ vanishes inside the propagation cone for any $y\in M$.

On the other hand, if $a_{\alpha\beta}$ are constant, the equation (1.2) is equivalent to an ordinary Euler-Poisson-Darboux type equation. It is then well-known that the equation is a HDE for certain integral values of λ. (see e.g. [1].)

In this note we shall make a remark that, under a certain condition, the same is true even when $a_{\alpha\beta}$ are not constant.

Theorem 1.1. *Suppose that ∇L is independent of x^1 when con-*

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sidered as a vector field on M by the obvious identification of $T(M)$ *with* $T^*(M)$. *Then the equation* (1.2) *becomes a HDE if and only if n is even and* λ *is an integer satisfying* $-(n/2)+2\leqq\lambda\leqq(n/2)-1$.

2. **Outline of the proof.** One way, perhaps the easiest, of proving the theorem is to construct the so-called Hadamard expansion of the fundamental solution $E_\lambda(x, y)$ to the equation (1.2). This can be written as

$$E_\lambda(x, y)=\sum_{\nu=0}^{\infty} U_\lambda^{(\nu)}(x, y)\chi_{\nu+1-n/2}(\Gamma(x, y)), \tag{2.1}$$

where $U_\lambda^{(\nu)}(x, y)\in C^\infty(M\times M)$. $\Gamma(x, y)$ is defined as the square of geodesic distance between the two points x, y of M. (If the metric (1.1) is replaced by the ordinary Minkowskian metric, we have $\Gamma(x, y)=(x_1-y_1)^2-\sum_2^n(x_j-y_j)^2$.) The functions χ_q are defined as follows; when Re $q>-1$,

$$\chi_q(t)=\begin{cases} t^q/\Gamma(q+1), & t>0 \\ 0, & t\leqq 0. \end{cases}$$

It is then easy to see that $\chi_q(t)$ can be continued to a distribution-valued entire function of q. We have, for instance,

$$\chi_{-q}(t)=\delta^{(q-1)}(t), \qquad q=1, 2, \cdots. \tag{2.2}$$

One need to be careful to define $\chi_q(\Gamma(x, y))$ since $\Gamma(x, y)$ has the double zero at $x=y$. We refer the precise definition to Delache and Leray [1] and here only note that $\chi_q(\Gamma)$ can be defined in such a way that its support is contained in the future propagation cone at y.

The functions $U_\lambda^{(\nu)}$ are successively determined by the following transport equations;

$$\begin{aligned} 2\langle\nabla\Gamma, \nabla U_\lambda^{(\nu)}\rangle+(\Box_M+\langle 2\nabla\log(\sigma L^\lambda), \nabla\Gamma\rangle+4\nu-2n)U_\lambda^{(\nu)} \\ =-PU_\lambda^{(\nu-1)}, \quad \nu=0, 1, \cdots, \quad (U^{(-1)}=0). \end{aligned} \tag{2.3}$$

To compute the solutions $U^{(\nu)}$ explicitly one need to know the exact form of geodesic flows (or the bicharacteristic flows). This is in fact made possible due to the rather simple form of our metric (1.1). (See [3] or [2, § 5.7].) We are then able to get

$$U_\lambda^{(0)}(x, y)=\frac{\pi^{1-n/2}}{2}\kappa(x^1, y^1)\left[\frac{L(x)}{L(y)}\right]^{-\lambda} \tag{2.4}$$

where

$$\kappa=\frac{|\det(\partial^2\Gamma/\partial x^i\partial x^j)|^{1/2}}{2^{n/2}|\det a_{\alpha\beta}(x^1)\det a_{\alpha\beta}(y^1)|^{1/4}}\cdot\frac{\sigma(y^1)}{\sigma(x^1)}.$$

The main argument in Günther [3] was to calculate $\Gamma(x, y)$ exactly in order to prove that $\det(\partial^2\Gamma/\partial x^i\partial x^j)$ and therefore $\kappa(x, y)$ depend only on (x^1, y^1). Once this proved and, if $\lambda=0$, we have $PU_0^{(0)}=0$. Then (2.3) shows that one can choose $U_0^{(\nu)}=0$, $\nu=1, 2, \cdots$, since the right-hand side vanishes. This leads to

$$E_0(x, y)=U_0^{(0)}(x^1, y^1)\chi_{1-n/2}(\Gamma),$$

which, combined with (2.2), shows that Huygens' principle is valid for $n=4, 6, \cdots$. For n odd, since supp $\chi_{1-n/2}(\Gamma)$ contains the inside of the propagation cone, Huygens' principle can not be expected.

In case of general λ, further computation shows that

$$(2.5)\qquad U_\lambda^{(\nu)}=\left(-\frac{1}{4}\right)^\nu \frac{\pi^{1-n/2}\kappa(x^1, y^1)}{2\nu\,!} \frac{L(x)^{-\lambda-\nu}}{L(y)^{-\lambda+\nu}}|\nabla L|^{2\nu} \prod_0^{\nu-1}(\lambda+k)(\lambda-k-1),$$
$$\nu=0, 1, \cdots.$$

In order for Huygens' principle to hold, it is necessary and sufficient that $U_\lambda^{(\nu)}$ vanishes for $\nu \geqq (n/2)-1$ (n is supposed to be even). By (2.5), this is equivalent to $\prod_0^{\nu-1}(\lambda+k)(\lambda-k-1)=0$, $\nu \geqq (n/2)-1$. The statement of Theorem 1.1 follows immediately from this observation.

3. Some trivial transformations. The equation (1.2) can be transformed into an ordinary wave equation in a different pseudo-Riemannian manifold by so-called trivial transformations. We first note that $\sigma(x^1)$ can be eliminated by considering $\sigma P \sigma^{-1}$ instead of P. This allows us to put $\sigma=1$ from now on.

We shall now show that a certain conformal transformation can reduce the equation (1.2) into an ordinary wave equation. Let $M_\mu = \boldsymbol{R}^n \setminus \mathrm{Ker}\, L$ be the pseudo-Riemannian manifold with the metric

$$|L(x)|^\mu (ds)^2=\sum g_{ij}\, dx^i dx^j, \qquad \mu \in \boldsymbol{R},$$

where $(ds)^2$ is defined by (1.1). Let $\Box_\mu$ be the ordinary wave operator defined on M_μ; namely,

$$\Box_\mu u=|g|^{-1/2}\sum \partial_i(|g|^{1/2} g^{ij}\partial_j u), \qquad \partial_i=\partial/\partial x_i,$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) and $|g|=|\det g_{ij}|$. A simple calculation gives

$$\Box_\mu u=|L(x)|^{-\mu}\left(\Box_M-\frac{(n-2)}{2}\mu\sum a^{\alpha\beta}(\partial_\alpha \log L(x))\partial_\beta\right)u$$
$$=|L(x)|^{-\mu}\left(\Box_M+\frac{n-2}{2}\mu\langle \nabla \log L(x), \nabla\rangle\right)u.$$

This formula will provide a restatement of Theorem 1.1. By a Huygens manifold we shall mean a pseudo-Riemannian manifold N with a Lorentzian metric, in which the wave equation $\Box_N u=0$ becomes an HDE.

Corollary 3.1. *Let the assumption for M and L in Theorem 1.1 hold here again. Then M_μ is a Huygens manifold if and only if n is even and $(n-2)\mu/4$ is an integer satisfying $-(n/2)+2 \leqq (n-2)\mu/4 \leqq (n/2)-1$.*

References

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