# 45. On the Rank of Hasse-Witt Matrix*)

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1. Let $A$ be an algebraic function field of one variable with a perfect field $K$ of characteristic $p\neq 0$ as the exact constant field. Let $D$ be the $K$-module of differentials of $A$. Let $G, E^*$ and $R$ be the $K$-submodules of differentials of the first kind, of pseudo-exact differentials and of residue free differentials in $D$, respectively.

The following equality was proven by the author [2], and by Kunz [4] in the case where $K$ is algebraically closed:

$$\dim_K R/E^* = \dim_K G/G\cap E^*.$$

The author proved in [3] that this equality still holds true and the both dimensions are unchanged by any algebraic constant field extension of $A$ over $K$.

Let $M$ be the Hasse-Witt matrix (identified with the Cartier-Manin matrix) of $A$ over $K$ with respect to a basis of $G$. Then we shall show

**Proposition.** *We have* $\operatorname{rank}(M^{(p^{1-g})}\cdots M^{(p^{-1})}M)=\dim_K G/G\cap E^*$, *where $g>0$ is the genus of $A$ and each $M^{(p^{-j})}$ is the matrix of $p^{-j}$-th power raised elements of $M$.*

**Corollary 1.** *The $p$-rank of the null class group of $A\bar{K}$, the constant field extension of $A$ by the algebraic closure $\bar{K}$ over $K$, is equal to* $\dim_K G/G\cap E^*$.

**Corollary 2.** $M^{(p^{1-g})}\cdots M^{(p^{-1})}M=0$ *holds if and only if* $G\subseteqq E^*$.

**Corollary 3.** *We have rank* $(M^{(p^{1-g})}\cdots M^{(p^{-1})}M)=\dim_K R/E^*$.

2. Let $A^p$ be the subfield of $p$-power elements of $A$. If $x$ is in $A\setminus A^p$, then $\{1, x, \cdots, x^{p-1}\}$ is a basis of $A$ over $A^p$, and any $\omega$ of $D$ is representable in such form as

$$\omega=\sum_{j=0}^{p-1} a_j^p x^j dx.$$

Then the Cartier operator $C$ is defined by $C(\omega)=a_{p-1}dx$. The following properties are well-known (see [1]);

(1) $C$ is independent of a choice of $x$.

(2) $C(y_1^p\omega_1+y_2^p\omega_2)=y_1C(\omega_1)+y_2C(\omega_2)$ for $y_1, y_2\in A$ and $\omega_1, \omega_2\in D$.

(3) $C(\omega)$ is in $G$ if $\omega$ is in $G$.

Let us denote by $E_n$ the $K$-submodule of $\omega$ of $D$ with $C^n(\omega)=0$. $E_{n+1}\supseteqq E_n$ for every $n$ is evident. Let us define $E^*=\bigcup_{n=1}^{\infty} E_n$ and call the elements of $E^*$ *pseudo-exact differentials*. In particular, we call the elements of $E_1$ *exact differentials*.

*) Dedicated to Professor Kentaro Murata on his 60th birthday.

**3. The proof of the proposition.** Let us denote $P=G\cap E^*$, $P_n=G\cap E_n$ and $G=P\oplus Q$, where $Q$ is a complementary module of $P$ in $G$. Let the dimensions of $P, P_n$ and $Q$ be $m, m_n$ and $m_0=g-m$, respectively. If $r$ is the minimum integer of these $n$ with $P\subseteqq E_n$, then $r\leqq m$. In fact, there exists such $n$ that $P\subseteqq E_n$, since $\dim P\leqq g$. Moreover if $r>m$, we take $\omega$ of $P_r\backslash P_{r-1}$. Then $C^{r-1}(\omega)\neq 0$ and $C^r(\omega)=0$. So $\{\omega, C(\omega), \cdots, C^{r-1}(\omega)\}$ are linearly independent over $K$. Hence $\dim_K P>m$, and this is a contradiction. Let $\{\omega_1, \cdots, \omega_{m_0}\}$ be a basis of $Q$ over $K$ and let $\{\omega_{m_{n-1}+1}, \cdots, \omega_{m_n}\}$ be a basis of $P_n \bmod P_{n-1}$ over $K$. Then $\{\omega_1, \cdots, \omega_{m_0}; \omega_{m_0+1}, \cdots, \omega_{m_1}; \cdots; \omega_{m_{r-1}+1}, \cdots, \omega_{m_r}\}$ forms a basis of $G$ over $K$. Hence $m_r=g$. Call this basis $W$. We represent $C$ by a matrix over $K$, the Cartier-Manin matrix, with respect to $W$ (the column vector of $\omega_j$'s):

$$C(W)={}^t(C(\omega_1), \cdots, C(\omega_g))=M\,{}^t(\omega_1, \cdots, \omega_g)=MW$$

with $M$, a $g\times g$ matrix over $K$, where since $C(P_n)\subseteqq P_{n-1}$ for every $n$, $M$ is of form

$$M=\begin{bmatrix} A_0 & A_1 & \cdots & \cdots & A_r \\ 0 & 0_1 & 0 & \cdots & 0 \\ 0 & A_{21} & 0_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_{r1} & A_{r2}\cdots & A_{rr-1} & 0_r \end{bmatrix},$$

$A_0, A_i$ and $A_{ij}$ are $m_0\times m_0$, $m_0\times(m_i-m_{i-1})$ and $(m_i-m_{i-1})\times(m_j-m_{j-1})$ matrices over $K$, respectively. $0_i$ denotes a $(m_i-m_{i-1})\times(m_i-m_{i-1})$ zero matrix. Assume that $\sum_{i=1}^{m_0} a_i C^j(\omega_i)=0$ for each $j\geqq 0$ with $a_i\in K$, then $\sum_{i=1}^{m_0} a_i^{p^j}\omega_i\in P\cap Q$. Hence all $a_i=0$ and $\{C^j(\omega_1), \cdots, C^j(\omega_{m_0})\}$ are linearly independent over $K$. So $\operatorname{rank} A_0=m_0$. By repeating application of $C$ to $W$, we obtain

$$C^g(W)=\tilde{M}W, \qquad \text{where } \tilde{M}=M^{(p^{1-g})}\cdots M^{(p^{-1})}M.$$

Then

$$\tilde{M}=\begin{bmatrix} \tilde{A}_0 & \tilde{A}_1\cdots\tilde{A}_r \\ 0 & \cdots\cdots\ 0 \\ \vdots & \vdots \\ 0 & \cdots\cdots\ 0 \end{bmatrix} \quad \text{with} \quad \tilde{A}_0=A_0^{(p^{1-g})}\cdots A_0^{(p^{-1})}A_0.$$

Therefore $\operatorname{rank}\tilde{M}=\operatorname{rank} A_0=m_0$. This completes the proof.

**4. The proof of Corollary 1.** As stated above, $\dim_K G/G\cap E^*=\dim_{\bar{K}} G'/G'\cap E^{*\prime}$, and hence these dimensions are equal to $\operatorname{rank}(M'^{(p^{1-g})}\cdots M'^{(p^{-1})}M')$ by the proposition, where the prime $'$ means the corresponding notions in $A\bar{K}$ over $\bar{K}$. As well-known, this is the $p$-rank of the null class group of $A\bar{K}$.

The Corollaries 2 and 3 are obvious.

**Remark 1.** A change of basis of $G$ induces a transformation $\tilde{M}\to S^{(p^{-g})}\cdot\tilde{M}\cdot S^{-1}$ with a $g\times g$ regular matrix $S$ over $K$.

**Remark 2.** One may choose a basis of $G$ mod.$G \cap E^*$ consisting of logarithmic differentials [5], when $K$ is algebraically closed.

## References

[1] M. Eichler: Einführung in die Theorie der algebraischen Zahlen und Funktionen. Basel, Birkhäuser Verlag (1963).

[2] T. Kodama: Residuenfreie Differentiale und der Cartier-Operator algebraischer Funktionenkörper. Arch. Math., **22**, 271–274 (1971).

[3] ——: Residue free differentials and the Cartier operator for algebraic function fields of one variable. Can. J. Math., **24**, 905–914 (1972).

[4] E. Kunz: Einige Anwendung des Cartier-Operators. Arch. Math., **13**, 349–356 (1962).

[5] J. P. Serre: Sur la topologie des variétés algébriques en caractéristique $p$. Sympos. Internac. Topol. Alg., Mexico City, 24–53 (1956).

[6] H. Hasse und E. Witt: Zyklische unverzweigte Erweiterungskörper vom Primzahlgrad $p$ über einem algebraischen Funktionenkörper der Charakteristik $p$. Monatsh. Math., **43**, 477–492 (1936).