38. On Pluricanonical Maps for 3-Folds of General Type

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(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1984)

The purpose of this note is to outline our recent result on the pluricanonical maps for nonsingular projective 3-folds of general type. Details will be published elsewhere.

Let X be a nonsingular projective 3-dimensional variety over the complex number field, which is called a "3-fold" in short. The canonical divisor K_X is said to be "nef" if the intersection number $K_X \cdot C \geqq 0$ for any curve C on X. Moreover, K_X is said to be "big" if $\kappa(K_X, X) = \dim X$ (cf. Iitaka [6] and Reid [10]), i.e., if X is of general type. For any n with $h^0(X, \mathcal{O}_X(nK_X)) \neq 0$, we have the n-ple canonical linear system $|nK_X|$ and associated with this, we have the rational map $\Phi_{|nK_X|}$.

Main Theorem. *Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big.*

Then

(i) $\Phi_{|7K_X|}$ *is birational with the possible exceptions of*

a) $\chi(\mathcal{O}_X) = 0$ *and* $K_X^3 = 2$, *or*

b) $|3K_X|$ *is composed of pencils,*

i.e., $\dim \Phi_{|3K_X|}(X) = 1$,

(ii) $\Phi_{|nK_X|}$ *is birational for* $n \geqq 8$.

Corollary. *Assume further that K_X is ample. Then $\Phi_{|nK_X|}$ is birational for $n \geqq 7$.*

The hypothesis that K_X is ample is required only to derive the inequality

$$\chi(\mathcal{O}_X) < 0 \qquad \text{(cf. Yau [11]).}$$

There is a conjecture that this inequality holds even when K_X is nef and big. Therefore, once this conjecture is established, we will have a sharper result that $\Phi_{|nK_X|}$ is birational for $n \geqq 7$ whenever K_X is nef and big.

X. Benveniste announced in [2] the same result as our main theorem. But his proof is incomplete. Modifying his argument, we can complete the proof and get a better result when K_X is ample.

§ 1. The following theorem about a surface plays a crucial role in our proof of the main theorem. We replace the condition $h^0(S, \mathcal{O}_S(mR)) \geqq 7$ in Proposition 2-0 of Benveniste [1] by (*) below, which is weaker than the former.

Theorem. *Let S be a nonsingular projective surface, $R \in \operatorname{Pic} S$ a nef divisor on S, and m a positive integer which satisfy the following condition* (*).

(*) *Take arbitrary two distinct points $x_1, x_2 \in S$. Let $\pi : S'' \to S$ be the blowing-up at x_1 and x_2, $L_1 := \pi^{-1}(x_1)$ and $L_2 := \pi^{-1}(x_2)$. Then we have $|\pi^*(mR) - 2L_1 - 2L_2| \neq \phi$.*

Then $\Phi_{|K_S+mR|}$ is birational in the following two cases

(i) $R^2 \geqq 2$ *and* $m \geqq 3$,

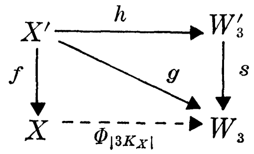
(ii) $R^2 = 1$ *and* $m \geqq 4$.

§2. Let X be a nonsingular projective 3-fold whose canonical divisor K_X is nef and big. Setting $W_n := \Phi_{|nK_X|}(X)$ for a positive integer n, we have the following assertions:

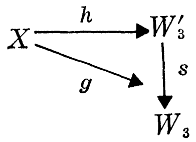
(i) $\dim W_n \geqq 2$ for $n \geqq 4$.

(ii) If $\dim W_3 = 1$, then one of the two cases α), β) holds.

First we consider the following commutative diagram and introduce some notations.



f is a succession of blowing-ups with nonsingular centers such that $g := \Phi_{|3K_X|} \cdot f$ is a morphism, the diagram



is the Stein factorization, $b := \deg(s)$, and S is a general fiber of h.

α) $b \cdot \{S \cdot f^*(K_X)^2\} = 2$, $\chi(\mathcal{O}_X) = 1$ and $K_X^3 = 6$,

β) $b = 1$, $S \cdot f^*(K_X)^2 = 1$ and S is a nonsingular projective surface of general type, so letting $\pi : S \to S_0$ be the morphism onto the minimal model S_0 of S, and K_0 the canonical divisor of S_0, we have $K_0^2 = 1$, and $\mathcal{O}_S(\pi^*(K_0)) \cong \mathcal{O}_S(f^*(K_X)|_S)$.

§3. Proof of the main Theorem. We show that $\Phi_{|nK_X|}$ is birational in each of the following four cases, which is sufficient by §2 and by hypotheses.

Case 1. $\dim W_3 \geqq 2$ and $n \geqq 8$,

Case 2. $\dim W_3 \geqq 2$, $\chi(\mathcal{O}_X) \neq 0$ or $K_X^3 \neq 2$, and $n = 7$,

Case 3. $\dim W_3 = 1$, β) and $n \geqq 8$,

Case 4. $\dim W_3 = 1$, α) and $n \geqq 8$,

where α) and β) are the cases described in §2.

In Cases 1, 2 and 4, the assertion can be obtained by using the theorem in §1. To get the result in Case 3, we use the fact that $\Phi_{|nK_S|}$

is birational for $n \geqq 5$ for any nonsingular projective surface of general type, which was obtained by Bombieri [3].

§4. **Proof of Corollary.** By the inequality $\chi(\mathcal{O}_X) \leqq -K_X^3/64 < 0$, we can show that $\dim W_3 \geqq 2$. So it is clear from Cases 1 and 2 in the proof of the main theorem.

References

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