34. A Shape of Eigenfunction of the Laplacian under Singular Variation of Domains. II

—The Neumann Boundary Condition—

By Shin OZAWA
Department of Mathematics, University of Tokyo

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Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary γ. Let B_ε be the ε-ball whose center is $w \in \Omega$. We put $\Omega_\varepsilon = \Omega \backslash \bar{B}_\varepsilon$. We consider the following eigenvalue problem:

$$(1) \qquad \begin{aligned} -\Delta_x u(x) &= \lambda(\varepsilon)u(x), & x \in \Omega_\varepsilon \\ u(x) &= 0, & x \in \gamma \\ \frac{\partial u}{\partial \nu}(x) &= 0, & x \in \partial B_\varepsilon, \end{aligned}$$

where $\partial/\partial\nu$ denotes the derivative along the inner normal vector at x with respect to the domain Ω_ε. Let $0 < \mu_1(\varepsilon) \leq \mu_2(\varepsilon) \leq \cdots$ be the eigenvalues of (1). Let $0 < \mu_1 \leq \mu_2 \leq \cdots$ be the eigenvalues of $-\Delta$ in Ω under the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities. Let $\{\varphi_j(\varepsilon)\}_{j=1}^\infty$ (resp. $\{\varphi_j\}_{j=1}^\infty$) be a complete orthonomal basis of $L^2(\Omega_\varepsilon)$ (resp. $L^2(\Omega)$) consisting of $-\Delta$ eigenfunctions of associated with $\{\mu_j(\varepsilon)\}_{j=1}^\infty$ (resp. $\{\mu_j\}_{j=1}^\infty$).

We assume that w is the origin of $\boldsymbol{R}^2$. We use the polar coordinates $z - w = (r\cos\theta, r\sin\theta)$. The aim of this note is to give the following:

Theorem 1. *Fix j. Assume that μ_j is a simple eigenvalue. Let ρ be an arbitrary fixed positive number. Then,*

$$(3) \qquad \|\varphi_j(\varepsilon) - t_\varepsilon \varphi_j\|_{L^\infty(\Omega_\varepsilon)} = O(\varepsilon^{1-\rho})$$

and

$$(4) \qquad \left(\left(\frac{\partial}{\partial\theta}(\varphi_j(\varepsilon))\right)(\varepsilon\cos\theta, \varepsilon\sin\theta)\right) = 2t_\varepsilon(\partial_{\vec{wz}}\varphi_j(w)|_{w=0}) + O(\varepsilon^{1-\rho})$$

hold, where $\partial_{\vec{wz}}\varphi_j(w)$ denotes the derivative of $\varphi_j(w)$ with respect to w along the vector $\overrightarrow{wz}$. Here

$$s_\varepsilon = \int_{\Omega_\varepsilon} (\varphi_j(\varepsilon))(x)\varphi_j(x)dx, \qquad t_\varepsilon = \operatorname{sgn} s_\varepsilon.$$

Remarks. The remainders in (3), (4) are not uniform with respect to j. We can prove that s_ε^2 tends to 1 as $\varepsilon \to 0$. The relationship between Theorem 1 and the following Theorem A in Ozawa [2] was discussed in Ozawa [2]. The Hadamard variational formula (see Garabedian-Schiffer [1]) plays an essential role in their relationship.

Theorem A. *Under the same assumptions of Theorem* 1

(5) $$\mu_j(\varepsilon)=\mu_j-(2\pi|\mathrm{grad}\,\varphi_j(w)|^2-\pi\mu_j\varphi_j(w)^2)\varepsilon^2+O(\varepsilon^3|\log\varepsilon|^2)$$

holds as ε *tends to zero.*

We here give an idea of our proof of Theorem 1. Let $G_\varepsilon(x, y)$ (resp. $G(x, y)$) be the Green's function of the Laplacian in Ω_ε (resp. Ω) under the Dirichlet condition on γ and the Neumann condition on ∂B_ε (resp. under the Dirichlet condition on γ). We put

$$p_\varepsilon(x, y\,;\tilde{w})=G(x, y)+\pi\varepsilon^2\Delta_{\tilde{w}}(G(x, \tilde{w})G(y, \tilde{w}))+(\pi/8)\varepsilon^4\Delta_{\tilde{w}}^2(G(x, \tilde{w})G(y, \tilde{w}))$$

for $x, y, \tilde{w}\in\Omega$, and we put $p_\varepsilon(x, y)=p_\varepsilon(x, y\,;w)$. The essential key to Theorem 1 lies in the fact that $p_\varepsilon(x, y)$ is a nice approximation of $G_\varepsilon(x, y)$ in Ω_ε as an integral kernel function. The kernel function $p_\varepsilon(x, y)$ was firstly introduced by Ozawa [2]. We use long and involved calculations using L^p-spaces. Details and further discussions will appear in Ozawa [6].

Additional remark. We here make an additional remark on the previous paper [4]. We follow the notations in [4]. Under the same assumption of [4; Theorem 1], we have the following formula which is more precise than that of [4]:

(6) $$\partial(\varphi_j(\varepsilon))(z)/\partial n_z^\varepsilon|_{z\in\partial B_\varepsilon}=-t_\varepsilon(\varepsilon^{-1}\varphi_j(w)-4\pi(\tau\varphi_j(w)-e_j(w))+3\frac{\partial}{\partial n_z^\varepsilon}\varphi_j(z))+O(\varepsilon^{1/2}),$$

where

$$t_\varepsilon=\mathrm{sgn}\int_{\Omega_\varepsilon}(\varphi_j(\varepsilon))(x)\varphi_j(x)dx$$

and $\partial/\partial n_z^\varepsilon$ denotes the derivative along the exterior normal direction with respect to Ω_ε. Here τ, $e_j(w)$ were the notations in Ozawa [3]. The formula (6) is discussed in Ozawa [5].

References

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