33. *On the Confluent Euler-Poisson-Darboux Equation and the Toda Equation*

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§ 1. **Summary.** The Toda equation with two time variables

(1.1) $$XY \log t_n = t_{n+1}t_{n-1}/t_n^2 \qquad \left(X=\frac{\partial}{\partial x},\ Y=\frac{\partial}{\partial y},\ t_n=t_n(x,y)\right)$$

can be solved using solutions of the confluent Euler-Poisson-Darboux equation

(1.2) $$(XY+xX+\alpha-n)u_n=0.$$

Rational solutions, confluent hypergeometric solutions and solutions which can be expressed by hypergeometric functions with two variables are obtained.

§ 2. **Bäcklund transformation of a separated solution.** As is shown in our previous work ([1])

(2.1) $$t_n=F(n)\exp((\alpha-n)xy)$$

where $F(n+1)F(n-1)/F(n)^2=\alpha-n$, $F(0)=F(1)=1$, satisfies the Toda equation (1.1).

(2.2) $$r_n=XY\log t_n=\alpha-n, \qquad s_n=Y\log t_{n-1}/t_n=x$$

satisfies

(2.3) $$Yr_n=r_n(s_n-s_{n+1}), \qquad Xs_n=r_{n-1}-r_n.$$

Put

(2.4) $$M_n=XY+s_{n+1}X+r_n=XY+xX+\alpha-n,$$
$$X_n=-r_n^{-1}X=(n-\alpha)^{-1}X, \qquad Y_n=Y+s_{n+1}=Y+x.$$

Define

(2.5) $$T=\{u_n=u_n(\alpha\,;x,y)\,;\,M_0u_0=0,\ u_{n+1}=Y_nu_n\ (n\geqslant 0),$$
$$u_{n-1}=X_nu_n\ (n\leqslant 0)\}$$

then we have

Theorem 2.1 (Bäcklund transformation). *If* $u_n\in T$ *then we have* $M_nu_n=0$, $u_{n+1}=Y_nu_n$, $u_{n-1}=X_nu_n$ $(n=0,\pm1,\pm2,\cdots)$ *and* $\tau_n=u_nt_n$ *satisfies the Toda equation* (1.1).

§ 3. **One-parameter groups on *T*.** We can obtain three linearly independent first order partial differential operators which commute with M_0 (modulo M_0).

Theorem 3.1. $\hat{X}=X+y$, Y *and* $Z=yY-xX$ *commute with* M_0.

We can construct three one-parameter groups of linear transformations and a finite group which keep T invariant.

Theorem 3.2 (Main theorem). *If $u_n \in T$ then*

(3.1) $\tilde{X}(\lambda)u_n(x, y)=e^{\lambda y}u_n(x+\lambda, y),\qquad \tilde{Y}(\mu)u_n(x, y)=u_n(x, y+\mu),$
$\tilde{Z}_n(\nu)u_n(x, y)=e^{n\nu}u_n(e^{-\nu}x, e^{\nu}y),$

(3.2) $Ru_n(\alpha; x, y)=(-1)^n(1-\alpha)_n e^{-xy}u_{-n}(1-\alpha, y, -x)$

belong to T. $\tilde{X}(\lambda)$, $\tilde{Y}(\mu)$ and $\tilde{Z}_n(\nu)$ are one-parameter groups of linear transformations with generators $\hat{X}, Y$ and $Z_n=yY-xX+n$ respectively. Each of these one-parameter groups and their corresponding generators keep $\ker M_n$ *invariant.* $\{R^4=\text{id.}, R, R^2, R^3\}$ *is a finite group.*

We can show the following commutation relations.

Theorem 3.3 (Commutation relations). *For any values of complex numbers λ, μ and ν we have*

(3.3) $\tilde{X}(\lambda)\tilde{Y}(\mu)=e^{-\lambda\mu}\tilde{Y}(\mu)\tilde{X}(\lambda),\qquad \tilde{X}(\lambda)\tilde{Z}_n(\nu)=\tilde{Z}_n(\nu)\tilde{X}(e^{-\nu}\lambda),$
$\tilde{Y}(\mu)\tilde{Z}_n(\nu)=\tilde{Z}_n(\nu)\tilde{Y}(e^{\nu}\mu),$

(3.4) $\tilde{X}(\lambda)Y=(Y-\lambda)\tilde{X}(\lambda),\qquad \tilde{X}(\lambda)Z_n=(Z_n-\lambda\hat{X})\tilde{X}(\lambda),$
$\tilde{Y}(\mu)Z_n=(Z_n+\mu Y)\tilde{Y}(\mu),\qquad \tilde{Y}(\mu)\hat{X}=(\hat{X}+\mu)\tilde{Y}(\mu),$
$\tilde{Z}_n(\nu)\hat{X}=e^{\nu}\hat{X}\tilde{Z}_n(\nu),\qquad \tilde{Z}_n(\nu)Y=e^{-\nu}Y\tilde{Z}_n(\nu),$

(3.5) $\hat{X}Y=Y\hat{X}-1,\quad \hat{X}Z_n=(Z_n-1)\hat{X},\quad YZ_n=(Z_n+1)Y,$

(3.6) $\tilde{X}(\lambda)R=R\tilde{Y}(-\lambda),\quad \tilde{Y}(\mu)R=R\tilde{X}(\mu),\quad \tilde{Z}_n(\nu)R=R\tilde{Z}_n(-\nu),$

(3.7) $\hat{X}R=-RY,\quad YR=R\hat{X},\quad Z_nR=-RZ_n.$

§ 4. Eigenfunctions. Eigenfunctions of Z_n are given by confluent hypergeometric functions ${}_1F_1(\alpha, \beta; z)$.

Theorem 4.1. *Dimension of the vector space* $T\cap\{u_n\in\ker(Z_n-\beta)\}$ *is 2. Its bases are given by*

(4.1) $f_n(\alpha, \beta; x, y)=A_n(x){}_1F_1(\alpha-\beta, n+1-\beta; -xy)$
$=A_n(x)e^{-xy}{}_1F_1(n+1-\alpha, n+1-\beta; xy),$
$g_n(\alpha, \beta; x, y)=Rf_n(\alpha, -\beta; x, y)$
$=B_n(y){}_1F_1(\alpha-n, 1+\beta-n; -xy)$
$=B_n(y)e^{-xy}{}_1F_1(1+\beta-\alpha, 1+\beta-n; xy),$

where $A_n(x)=(1-\alpha)_n x^{n-\beta}/(1-\beta)_n$, $B_n(y)=(-1)^n(-\beta)_n y^{\beta-n}$. *For* $j=0, 1, 2, \cdots$ *we have the following relations.*

(4.2) $(-\hat{X})^j f_n(\alpha, \beta; x, y)=(\beta)_j f_n(\alpha, \beta+j; x, y),$
$(-Y)^j f_n(\alpha, \beta; x, y)=((\alpha-\beta)_j/(1-\beta)_j)f_n(\alpha, \beta-j; x, y),$
$\hat{X}^j g_n(\alpha, \beta; x, y)=((1+\beta-\alpha)_j/(1+\beta)_j)g_n(\alpha, \beta+j; x, y),$
$(-Y)^j g_n(\alpha, \beta; x, y)=(-\beta)_j g_n(\alpha, \beta-j; x, y).$

Theorem 4.2. *A base of 1-dimensional vector space*

$$T\cap\{u_n\in\ker Y\}=T\cap\{u_n\in\ker Y\cap\ker(Z_n-\alpha)\}$$

is given by

(4.3) $p_n=x^{n-\alpha}=f_n(\alpha, \alpha; x, y).$

(4.4) $\tilde{X}(\lambda)p_n=e^{\lambda y}(x+\lambda)^{n-\alpha}$

is a base of 1-dimensional vector space

$$T\cap\{u_n\in\ker(Y-\lambda)\}=T\cap\{u_n\in\ker(Y-\lambda)\}\cap\ker(Z_n-\alpha-\lambda\hat{X}).$$

(4.5) $q_n=Rp_n=(-1)^n(1-\alpha)_n e^{-xy}y^{\alpha-1-n}=g_n(\alpha, \alpha-1; x, y)$

is a base of 1-dimensional vector space

$$T\cap\{u_n\in\ker\hat{X}\}=T\cap\{u_n\in\ker\hat{X}\cap\ker(Z_n+1-\alpha)\}.$$

(4.6) $\tilde{Y}(\mu)q_n=R\tilde{X}(\mu)p_n=(-1)^n(1-\alpha)_n e^{-x(y+\mu)}(y+\mu)^{\alpha-1-n}$

is a base of 1-dimensional vector space

$$T\cap\{u_n\in\ker(\hat{X}+\mu)\}=T\cap\{u_n\in\ker(\hat{X}+\mu)\cap\ker(Z_n+1-\alpha+\mu Y)\}.$$

§5. Rational solutions. Theorem 5.1 (Rational solutions).

(5.1) $P_{n,k}=\hat{X}^k p_n/p_n=(\alpha-n)_k(-x)^{-k}{}_1F_1(-k,n+1-\alpha-k;-xy)$

is a homogeneous polynomial of order k in (x^{-1},y).

(5.2) $\rho_n=\alpha-n+XY\log P_{n,k}=(\alpha-n)P_{n+1,k}P_{n-1,k}/P^2_{n,k},$

$\sigma_n=x+Y\log P_{n-1,k}/P_{n,k}$

is a rational solution of the Toda equation (2.3).

$\tilde{P}_{n,k}=Z^k_n\tilde{X}(\lambda)p_n/\tilde{X}(\lambda)p_n,\quad Q_{n,k}=Y^k q_n/q_n,\quad \tilde{Q}_{n,k}=Z^k_n\tilde{Y}(\mu)q_n/\tilde{Y}(\mu)q_n$

are essentially polynomials and give rational solutions of the Toda equation.

§6. Hypergeometric solutions. By eigenfunction expansion we can construct various solutions of the Toda equation. If

(6.1) $u_n=\sum_{j=0}^{\infty}a_j f_n(\alpha,\beta+\varepsilon j;x,y)$ (ε is an integer)

converges then it belongs to T. If we choose ε and a_j suitably then we can express u_n by hypergeometric functions with two variables of order two which we can find in Horn's list ([2]).

Theorem 6.1 (Hypergeometric solutions). *$\alpha,\beta,\delta,\alpha'$ and β' represent arbitrary constants.*

(i) $\varepsilon=1,\ a_j=(\alpha')_j(\beta)_j/(\delta)_j j!,$

(6.2) $u_n=A_n(x)e^{-xy}\sum_{j,k}\frac{(\beta-n)_{j-k}(\alpha')_j(n+1-\alpha)_k}{(\delta)_j j!\,k!}x^{-j}(-xy)^k$

$=A_n(x)e^{-xy}H_2(\beta-n,\alpha',n+1-\alpha,\delta;x^{-1},-xy)$

$={}_1F_1(\alpha',\delta;-\hat{X})f_n(\alpha,\beta;x,y),$

(6.3) $u_n(x,0)=A_n(x){}_2F_1(\alpha',\beta-n,\delta;x^{-1}).$

If we choose $\delta=1+\beta-\alpha$ then we have

(6.4) $u_n=A_n(x)\sum_{j,k}\frac{(\alpha')_j(\beta-n)_{j-k}}{(1+\beta-\alpha)_{j-k}j!\,k!}x^{-j}(-xy)^k$

$=A_n(x)\Gamma_1(\alpha',\alpha-\beta,\beta-n;-x^{-1},xy)$

$={}_1F_1(\alpha',1+\beta-\alpha;-\hat{X})f_n(\alpha,\beta;x,y),$

(6.5) $u_n(x,0)=A_n(x){}_2F_1(\alpha',\beta-n,1+\beta-\alpha;x^{-1}).$

(ii) $\varepsilon=1,\ a_j=(\beta)_j/(\delta)_j j!,$

(6.6) $u_n=A_n(x)e^{-xy}H_4(\beta-n,n+1-\alpha,\delta;x^{-1},-xy)$

$={}_0F_1(\delta;-\hat{X})f_n(\alpha,\beta;x,y),$

(6.7) $u_n(x,0)=A_n(x){}_1F_1(\beta-n,\delta;x^{-1}).$

If we choose $\delta=1+\beta-\alpha$ then we have

(6.8) $u_n=A_n(x)\Gamma_2(a-\beta,\beta-n;-x^{-1},xy)$

$={}_0F_1(1+\beta-\alpha;-\hat{X})f_n(\alpha,\beta;x,y),$

(6.9) $u_n(x,0)=A_n(x){}_1F_1(\beta-n,1+\beta-\alpha;x^{-1}).$

$A_n(x)e^{-xy}H_9(\beta-n,n+1-\alpha,\delta;x^{-2},-xy),$

$$A_n(x)e^{-xy}\Xi_1(\alpha', n+1-\alpha, \beta', n+1-\beta; x, xy),$$
$$A_n(x)\Phi_1(\alpha-\beta, \beta', n+1-\beta; -xy, x),$$
$$A_n(x)e^{-xy}\Phi_2(\beta', n+1-\alpha, n+1-\beta; x, xy),$$
$$A_n(x)e^{-xy}\Phi_3(n+1-\alpha, n+1-\beta; xy, x)$$

also belong to T. We used Pochhammer's notation.

$$_pF_q(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q; z)=\sum_{j=0}^{\infty}\frac{(\alpha_1)_j\cdots(\alpha_p)_j}{(\beta_1)_j\cdots(\beta_q)_j j!}z^j,$$
$$(a)_j=\Gamma(j+a)/\Gamma(a).$$

References

[1] Y. Kametaka: On the telegraph equation and the Toda equation. Proc. Japan Acad., **60**A, 79–81 (1984).
[2] A. Erdelyi *et al.*: Higher Transcendental Functions. vol. 1, pp. 224–227, McGraw-Hill (1953).