32. On the Growth of Meromorphic Solutions of an Algebraic Differential Equation

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(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1984)

1. Introduction. In 1933 Yosida ([14]) applied the Nevanlinna theory of meromorphic functions to differential equations in the complex plane for the first time and generalized a Malmquist's theorem ([7]).

Theorem of Yosida. *If the differential equation*

(1) $(w')^m = R(z, w)$, *R rational in z, w and m a positive integer, possesses a transcendental meromorphic solution* $w = w(z)$ *in the complex plane, then* $R(z, w)$ *must be a polynomial in w of degree at most 2m. Further, if w(z) has only a finite number of poles, the degree is at most m.*

Later various mathematicians studied differential equations in the complex plane with the aid of Nevanlinna theory (see the references in [1], [13]) and many generalizations of this theorem have been obtained by several authors ([2], [5], [6], [11], [12], etc.).

In this paper we shall consider a general differential equation studied in [2], [6], [11] and [12]. We denote by $\mathcal{M}$ the set of meromorphic functions in the complex plane and by $\mathcal{L}$ the set of $E \subset [0, \infty)$ for which means $E < \infty$. Further, the term "meromorphic" will mean meromorphic in the complex plane.

Let P be a polynomial of $w, w', \cdots, w^{(n)}$ $(n \geqq 1)$ with coefficients in $\mathcal{M}$:

$$P(z, w, w', \cdots, w^{(n)}) = \sum_{\lambda \in I} c_\lambda(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n},$$

where $c_\lambda \in \mathcal{M}$ and I is a finite set of multi-indices $\lambda = (i_0, i_1, \cdots, i_n)$ for which $c_\lambda \neq 0$ and $i_0, i_1, \cdots, i_n$ are non-negative integers, and let $A(z, w)$, $B(z, w)$ be polynomials in w with coefficients in $\mathcal{M}$ and mutually prime in $\mathcal{M}$:

$$A(z, w) = \sum_{j=0}^{p} a_j(z) w^j, \qquad B(z, w) = \sum_{k=0}^{q} b_k(z) w^k,$$

where $a_j, b_k \in \mathcal{M}$ such that $a_p \cdot b_q \neq 0$.

We shall consider the differential equation

$$(2) \qquad P(z, w, w', \cdots, w^{(n)}) = A(z, w)/B(z, w).$$

We put

$$\Delta = \max_{\lambda \in I} (i_0 + 2i_1 + \cdots + (n+1)i_n),$$
$$d = \max_{\lambda \in I} (i_0 + i_1 + \cdots + i_n),$$
$$\Delta_o = \max_{\lambda \in I} (i_1 + 2i_2 + \cdots + ni_n).$$

A meromorphic solution $w=w(z)$ of (2) is said to be admissible when it satisfies

$$T(r,f)=o(T(r,w)) \qquad (r\to\infty,\ r\notin E\in\mathcal{L})$$

for all coefficients $f=a_j, b_k$ and c_λ in (2).

As a generalization of Theorem of Yosida cited above, Gackstatter and Laine ([2]) and Steimetz ([11]) proved the following:

"*If the differential equation* (2) *possesses an admissible solution* $w=w(z)$, *then* $q=0$ *and* $p\leqq \Delta$. *Further, if* $N(r,w)=o(T(r,w))$ $(r\to\infty,$ $r\notin E\in\mathcal{L})$, *then* $p\leqq d$."

Another proof is given in § 6 of [1].

The purpose of this paper is to give a more precise result than this. We shall make an essential use of inequalities in [8] and [9]. It is assumed that the reader is familiar with the notation of Nevanlinna theory (see [3], [4] or [10]).

2. Lemmas. We shall give some lemmas for later use.

For nonconstant $f\in\mathcal{M}$, we denote by $S_o(r,f)$ any quantity satisfying

$$S_o(r,f)=\begin{cases} O(1)\ (r\to\infty), \text{ when } f \text{ is rational},\\ O(\log r)\ (r\to\infty), \text{ when } f \text{ is transcendental of finite order},\\ O(\log rT(r,f))\ (r\to\infty,\ r\notin E\in\mathcal{L}), \text{ when } f \text{ is of infinite order}.\end{cases}$$

Lemma 1. *Let f be nonconstant meromorphic, then*

$m(r,f^{(i)}/f)=S_o(r,f)$ $(i=1,2,\cdots)$ (see [3], [4] or [10]).

Lemma 2. *Let $f, d_j\in\mathcal{M}$ and $A(z,w)$, $B(z,w)$ be as in § 1. Then,*

(i) $T(r,\sum_{j=0}^{t} d_j f^j)\leqq tT(r,f)+\sum_{j=0}^{t} T(r,d_j)+O(1)$ (see [3], p. 46).

(ii) *If $A(z,f(z))\not\equiv 0$ and $B(z,f(z))\not\equiv 0$,*

$$T(r,A(z,f)/B(z,f))=\max(p,q)T(r,f) +O\Big(\sum_{j=0}^{p} T(r,a_j)+\sum_{k=0}^{q} T(r,b_k)\Big)+O(1)$$

([8]).

Lemma 3. *Let P, Δ, d and Δ_o be as in § 1, $w=w(z)$ nonconstant meromorphic and $\alpha\in C$. Then,*

(i) $T(r,P/(w-\alpha)^{\Delta})\leqq \Delta T(r,w)+\sum_{\lambda\in I} T(r,c_\lambda)+S_o(r,w)$;

(ii) $T(r,P/(w-\alpha)^{d})\leqq dT(r,w)+\Delta_o\overline{N}(r,w)+\sum_{\lambda\in I} T(r,c_\lambda)+S_o(r,w)$.

We can prove this lemma without difficulty applying the method used in [9] and using Lemma 1.

3. Theorem. We use the same notation as in §§ 1–2.

Theorem. *Let $w=w(z)$ be any nonconstant meromorphic solution of the differential equation* (2).

(I) *When $q\neq 0$ or $p>\Delta$,*

$$\max(q,p-\Delta)T(r,w)\leqq \sum_{\lambda\in I} T(r,c_\lambda) +O\Big(\sum_{j=0}^{p} T(r,a_j)+\sum_{k=0}^{q} T(r,b_k)\Big)+S_O(r,w).$$

(II) *When $q\neq 0$ or $p>d$,*

$$\max(q,p-d)T(r,w)\leqq \Delta_o\overline{N}(r,w)+\sum_{\lambda\in I} T(r,c_\lambda) +O\Big(\sum_{j=0}^{p} T(r,a_j)+\sum_{k=0}^{q} T(r,b_k)\Big)+S_o(r,w).$$

Proof. If $A(z,w(z))\equiv 0$, then from

$$a_p w^p = -(a_{p-1}w^{p-1} + \cdots + a_0),$$

we have by Lemma 2(i)

$$T(r, w) \leqq \sum_{j=0}^{p} T(r, a_j) + O(1).$$

This inequality is contained in any case of this theorem, so that we have only to prove this theorem when $A(z, w(z)) \not\equiv 0$.

Let α be a constant such that

$$A(z, \alpha) = a_0 + \alpha a_1 + \cdots + \alpha^p a_p \not\equiv 0.$$

This is possible as $a_p \not\equiv 0$.

(I) Substituting $w = w(z)$ in (2) and dividing by $(w(z) - \alpha)^{\Delta}$, we have the relation

$$P(z, w, w', \cdots, w^{(n)})/(w-\alpha)^{\Delta} = A(z, w)/(w-\alpha)^{\Delta} B(z, w).$$

Note that $A(z, w)$ and $(w-\alpha)^{\Delta} B(z, w)$ are mutually prime in $\mathcal{M}$ because of the choice of α. From this relation, we obtain the following inequality by Lemmas 2(ii) and 3(i):

$$\Delta T(r, w) + \sum_{\lambda \in I} T(r, c_\lambda) + S_0(r, w) \geqq \max(p, q+\Delta) T(r, w)$$
$$+ O\left(\sum_{j=0}^{p} T(r, a_j) + \sum_{k=0}^{q} T(r, b_k)\right) + O(1),$$

which reduces to the desired inequality.

(II) Substituting $w = w(z)$ in (2) and dividing by $(w(z) - \alpha)^d$, we have the relation

$$P(z, w, w', \cdots, w^{(n)})/(w-\alpha)^d = A(z, w)/(w-\alpha)^d B(z, w),$$

from which we can easily obtain the desired inequality with the aid of Lemmas 2(ii) and 3(ii) as in the case of (I) as $A(z, w)$ and $(w-\alpha)^d B(z, w)$ are mutually prime in $\mathcal{M}$.

Remark. This theorem contains the result in [12].

References

[1] A. E. Eremenko: Meromorphic solutions of algebraic differential equations. Russian Math. Surveys, **37**-4, 61–95 (1982).

[2] F. Gackstatter and I. Laine: Zur Theorie der gewöhnlichen Differentialgleichungen im Komplexen. Ann. Polo. Math., **38**, 259–287 (1980).

[3] A. A. Gol'dberg and I. V. Ostrowskii: The Distribution of Values of Meromorphic Functions. Nauka, Moskow (1970) (in Russian).

[4] W. K. Hayman: Meromorphic Functions. Oxford Univ. Press (1964).

[5] I. Laine: On the behavior of the solutions of some first-order differential equations. Ann. Acad. Sci. Fenn., Ser. AI, **497**, 1–26 (1971).

[6] ——: Admissible solutions of some generalized algebraic differential equations. Publ. Univ. Joensuu, Ser. B, **10**, 1–6 (1974).

[7] J. Malmquist: Sur les fonctions à un nombre fini de branches définies par les équations différentielles du premier ordre. Acta Math., **36**, 297–343 (1913).

[8] A. Z. Mokhon'ko: On the Nevanlinna characteristics of some meromorphic functions. Theory of Functions. Funct. Anal. Appl., **14**, 83–87 (1971).

[9] A. Z. Mokhon'ko and V. D. Mokhon'ko: Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their applications to differential equations. Sib. Math. J., **15**-6, 921–934 (1974).

[10] R. Nevanlinna: Le théorème de Picard-Borel et la théorie des fonctions méromorphes. Gauthier-Villars, Paris (1929).
[11] N. Steimetz: Bemerkung zu einem Satz von Yosida. Lect. Notes in Math., vol. 747, Springer, pp. 369–377 (1979).
[12] SH. Strelitz: On meromorphic solutions of algebraic differential equations. Trans. Amer. Math. So., **258**-2, 431–440 (1980).
[13] H. Wittich: Anwendungen der Wertverteilungslehre auf gewöhnliche Differentialgleichungen. Ann. Acad. Sci. Fenn., Ser. AI Mathematica, **7**, 89–97 (1982).
[14] K. Yosida: A generalization of Malmquist's theorem. Japan. J. Math., **9**, 253–256 (1933).