31. A Varifold Solution of the Nonlinear Wave Equation of a Membrane

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§1. **Introduction.** Let U be a bounded domain in $\mathbf{R}^n$ with the boundary ∂U which is a Lipschitz manifold. Let $D_j=\partial/\partial x_j$, $j=1, 2, \cdots, n$, and $D_t=\partial/\partial t$. Then the nonlinear wave equation we shall consider is as follows:

(1) $$D_t^2 u(t, x)-\sum_{j=1}^n D_j\{D_j u(t, x)(1+|Du(t, x)|^2)^{-1/2}\}=0.$$

(2) $$u(t, x)=u_0(x), \qquad D_t u(0, x)=u_1(x).$$

(3) $$u(t, x)=g(x) \qquad \text{for } x \text{ in } \partial U.$$

The global existence of a weak solution of the above equation is not yet proved in general. (See §2 below for the definition of the weak solution.) In this paper, we shall try to treat the equations (1)–(3) by virtue of the theory of varifolds (cf. [1] and [2]) and prove the global existence of a varifold solution of them. Although a varifold solution is quite a weak notion, the varifold solution existence of which we can prove satisfies a generalized energy conservation law and is a solution of a problem of calculus of variation, which is a natural generalization of Hamilton's principle:

(4) $$\delta\int_0^T dt\int_U \left\{\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2-\sqrt{1+|Du|^2}\right\}dx=0.$$

Proofs of the results in this paper will be published elsewhere.

§2. **A weak solution.** We shall denote by $BV(U)$ the space of all functions of bounded variation in U, that is, $u\in BV(U)$ if and only if $u\in L^1(U)$ and its gradient $Du=(D_1u, D_2u, \cdots, D_nu)$ is a vector valued Radon measure (cf. [3]). We denote its total variation by $|Du|$. The Sobolev space $H^1(U)$ of order 1 is contained in $BV(U)$. If $u\in BV(U)$ then its trace γu from the interior of U is a function in $L^1(\partial U)$. For u in $BV(U)$, the set $E_u=\{(x, y)\in U\times\mathbf{R}\,|\,y<u(x)\}$ is a Caccioppoli subset of $\mathbf{R}^{n+1}$. At each point (x, y) of the reduced boundary ∂^*E_u of E_u, we can define the exterior unit normal $\nu(x, y)=(\nu_1(x, y), \nu_2(x, y), \cdots, \nu_n(x, y), \nu_{n+1}(x, y))$ to E_u. The characteristic function χ_E of E_u is of bounded variation. $|D\chi_E|$ denotes the total variation of the gradient $D\chi_E$.

Definition 2.1. Assume that $u_0\in H^1(U)$, $u_1\in L^2(U)$ and that g is the trace of some function in $BV(U)$. Then a function $u(t, x)\in L^1_{\mathrm{loc}}(\mathbf{R}\times U)$ is a weak solution of the equations (1), (2) and (3) if the following conditions hold:

(i) For each $t \in \mathrm{R}$, $u(t,x)$ is a function of bounded variation with respect to x such that $\gamma u = g$.

(ii) For each $\psi(t,x) \in C^2([0,T); C_0(U)) \cap C([0,T); C_0^2(U))$, we have

$$\int_U D_t\psi(0,x)u_0(x)dx - \int_U \psi(0,x)u_1(x)dx = \int_0^T dt \int_{U\times\mathrm{R}} \Big\{ -D_t^2\psi(t,x)u(t,x) + \sum_{j=1}^n D_j\psi(t,x)\nu_j(t;x,y) \Big\} \nu_{n+1}(t;x,y)\,|D\chi_E|, \tag{2.1}$$

where χ_E denotes the characteristic function of the set $E_u = \{(x,y) \in U\times\mathrm{R} \mid y < u(t,x)\}$ and $\nu(t;x,y) = (\nu_1(t;x,y), \nu_2(t;x,y), \cdots, \nu_{n+1}(t;x,y))$ is the exterior unit normal to E_u.

In the following we shall consider the case $g=0$.

§3. A varifold solution. We shall define the notion of a varifold solution of the equation (1). Let $G = G(n+1, n)$ be the Grassmann manifold of all n-dimensional vector subspaces of R^{n+1}. Let $S \in G$ be an n-dimensional vector subspace of R^{n+1}. Then we denote by $\nu(S) = (\nu_1(S), \nu_2(S), \cdots, \nu_{n+1}(S))$ its unit normal to S. We choose $\nu(S)$ so that $\nu_{n+1}(S) \geqq 0$. If $\nu_{n+1}(S)=0$, then $\nu(S)$ is not defined. However $\nu_{n+1}(S)$ and $\nu_{n+1}(S)\nu_j(S)$, $j=1,2,\cdots,n$, can be extended as continuous functions defined on the whole of G. The space $U\times\mathrm{R}\times G$ is called the Grassmann bundle of $U\times\mathrm{R}$ and denoted by $G(U\times\mathrm{R})$. A point of $G(U\times\mathrm{R})$ is denoted by (x,y,S), where $x\in U$, $y \in \mathrm{R}$ and $S \in G$.

A varifold (more precisely, an n-varifold) $V(x,y,S)$ is a positive Radon measure defined on the Grassmann bundle $G(U\times\mathrm{R})$. (See Allard [1] for detailed discussions.) Using this notion, we can give the following

Definition 3.1. A varifold $V(t;x,y,S)$ depending on the parameter $t\in[0,T)$ is called a *varifold solution* of the nonlinear equation (1) if, for each $\psi \in C^2([0,T); C_0(U)) \cap C([0,T); C_0^2(U))$, we have

$$\int_0^T dt \int_{G(U\times\mathrm{R})} \Big\{ -D_t^2\psi(t,x)y\nu_{n+1}(S) + \sum_{j=1}^n D_j\psi(t,x)\nu_j(S)\nu_{n+1}(S) \Big\} dV(t;x,y,S) = \int_U D_t\psi(0,x)u_0(x)dx - \int_U \psi(0,x)u_1(x)dx.$$

Remark 3.2. If a varifold solution $V(t;x,y,S)$ of (1) can be identified with a graph $\{y=u(t,x)\}$ of a function $u(t,x)$ of class C^1, then $u(t,x)$ is a weak solution of (1) in the sense of Definition 2.1.

We can prove the following theorem by the Galerkin method.

Theorem 1. *Assume that* $u_0(x) \in BV(U)$ *and that* $u_1(x) \in L^2(U)$. *Then for an arbitrary* $T>0$, *there exists a varifold solution* $V(t;x,y,S)$ *of the equation* (1) *for* $t\in[0,T)$.

The following problem naturally arises.

Problem. Can one identify the varifold solution $V(t; x, y, S)$ of Theorem 1 with a rectifiable set in $U \times \mathrm{R}$ for each t in $[0, T)$?

§ 4. The extremal property of a varifold solution.

Definition 4.1. Let $V(t; x, y, S)$ be a varifold depending on the parameter $t \in [0, T)$. For each t, we define a positive measure $\mu(t, x, y)$ on $U \times \mathrm{R}$ by the following equality: For any $\psi(x, y) \in C_0(U \times \mathrm{R})$,

$$\langle \psi, \mu \rangle = \int_{G(U \times \mathrm{R})} \psi(x, y) \nu_{n+1}(S) dV(t; x, y, S). \tag{4.1}$$

We call $\mu(t, x, y)$ *the mass distribution* of the membrane if $V(t; x, y, S)$ is a varifold solution of (1). Let $\Phi(S)$ be the characteristic function of the set $G_0 = \{S \in G \mid \nu_{n+1}(S) = 0\}$. Then we define the measure $B(t, x, y)$ on $U \times \mathrm{R}$ by

$$\langle \psi, B \rangle = \int_{G(U \times \mathrm{R})} \psi(x, y) \Phi(S) dV(t; x, y, S).$$

We call Spt B the *set of catastroph points* of the membrane.

Definition 4.2. Given a varifold $V(t; x, y, S)$ with the parameter t and a sphere $B_\rho(x)$ of the radius ρ with the center $x \in U$, we define

$$H_\rho(t, x) = \int_{B_\rho(x) \times \mathrm{R} \times G} y \nu_{n+1}(S) dV(t, z, y, S).$$

Let $|B_\rho(x)|$ stand for the volume of the sphere. Then the limit

$$v(t, x) = \lim_{\rho \to 0} \frac{H_\rho(t, x)}{|B_\rho(x)|}$$

exists almost every x in U. We call $v(t, x)$ the *position of the membrane* if $V(t; x, y, S)$ is a varifold solution of the equation (1).

Remark 4.3. The varifold solution $V(t; x, y, S)$ we constructed in Theorem 1 satisfies the *generalized energy conservation law*:

$$\int_U \frac{1}{2} |D_t v(t, x)|^2 dx + \int_{G(U \times \mathrm{R})} dV(t; x, y, S)$$
$$= \int_U \frac{1}{2} |u_1(x)|^2 dx + \int_U \sqrt{1 + |Du_0|^2} dx.$$

Using $v(t, x)$, we define the following *action integral*:

$$A(V) = \int_0^T dt \left\{ \int_U \frac{1}{2} |D_t v(t, x)|^2 dx - \int_{G(U \times \mathrm{R})} dV(t; x, y, S) \right\}.$$

Let $\psi(t, x)$ be a function in $C^2([0, T); C_0(U)) \cap C([0, T); C_0^2(U))$. Then we can define a one parameter group of translations:

$$\eta(\sigma): U \times \mathrm{R} \ni (x, y) \longrightarrow (x, y + \sigma \psi(t, x)) \in U \times \mathrm{R}.$$

This induces a map of a varifold V to another varifold $\eta(\sigma)_* V$. The extremal property of the varifold solution of the wave equation is stated as follows:

Theorem 2. *The varifold solution $V(t; x, y, S)$ of Theorem 1 of the wave equation (1) is an extremal of the action $A(V)$ with respect to the one parameter family of deformations $\eta(\sigma)_*$, that is,*

(4.2)
$$\frac{d}{d\sigma}A(\eta(\sigma)_*V)|_{\sigma=0}=0.$$

Remark 4.4. This is a *generalization of Hamilton's principle* (4).

References

[1] Allard, W. K.: On the first variation of a varifold. Ann. Math., 417–491 (1972).

[2] Almgren, F. J., Jr.: The theory of varifolds, A variational calculus in the large for the k-dimensional area integrand. Princeton (1965) (Mimeographed note).

[3] Giusti, E.: Minimal surfaces and functions of bounded variation. Notes on Pure Math. Australian National Univ., Canberra (1977).