138. *Deformations of Complements of Lines in* $\mathbf{P}^2$

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§1. **Introduction.** In this paper we shall study deformations of complements of lines in $\boldsymbol{P}^2$, based on the theory of logarithmic deformation introduced by Kawamata ([2]). The result is that the standard completion (see below) of complements of lines in $\boldsymbol{P}^2$ has smooth versal family of logarithmic deformations. This provides examples of surfaces of logarithmic general type with unobstructed deformations even though $H^2(X, \Theta(\log D)) \neq 0$.

Let $\Delta_1, \cdots, \Delta_n$ be projective lines on a complex projective plane $\boldsymbol{P}^2$, where $\Delta_i \neq \Delta_j$ for $i \neq j$, and let $\Delta = \bigcup_i \Delta_i$. We call $P \in \Delta$ a higher multiple point of Δ, if the multiplicity of Δ at P is greater than two. Let $P_1, \cdots, P_s$ be all the higher multiple points of Δ with respective multiplicities $\nu_1, \cdots, \nu_s$. Let $\mu_1, \cdots, \mu_n$ be the numbers of higher multiple points lying over $\Delta_1, \cdots, \Delta_n$, respectively. Blowing up $\boldsymbol{P}^2$ with center at $C = P_1 + \cdots + P_s$, we obtain a complete non-singular surface X and a birational morphism $\mu: X \to \boldsymbol{P}^2$. Let $E_j = \mu^{-1}(P_j)$, Δ^* the proper transform of Δ and D the set-theoretical inverse image of Δ, i.e. $D = \mu^{-1}(\Delta) = \Delta^* + \Sigma_j E_j$. Then D is a divisor on X with normal crossings. The non-singular triple $(X \backslash D, X, D)$ is called the standard completion of $\boldsymbol{P}^2 \backslash \Delta$ (cf. [1, p. 4]) and can be used as a substitute for the complement of lines Δ in $\boldsymbol{P}^2$.

For the definition of the family of logarithmic deformations of non-singular triple, we refer to [2].

Then we have the following

Theorem. (1) *For any choice of* Δ, *the non-singular triple* $\xi = (X \backslash D, X, D)$ *has no obstruction to logarithmic deformations.*

(2) *The numbers* $h^i = \dim H^i(X, \Theta(\log D))$ *are computed and classified according to the type* (cf. [1, Table]) *of* Δ *as following* Table I.

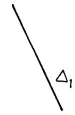
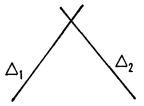
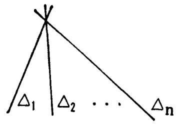
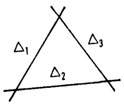
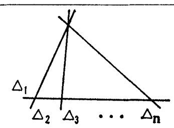
(3) *If* Δ *corresponds to the configurations of Pappus or Desargues* (Fig. 1), *then we get examples with* $H^2(X, \Theta(\log D)) \neq 0$.

(4) *There exists an infinite series of* Δ*'s of type* III *with* $H^1(X, \Theta(\log D)) = 0$.

In this paper we outline a proof of (1). For the details we refer to [3].

§2. **Unobstructedness of** $(X \backslash D, X, D)$. Let $(\hat{\mathcal{X}} \backslash \hat{\mathcal{D}}, \hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\pi}, \hat{B})$ be the versal family of logarithmic deformations of $(X \backslash D, X, D)$ con-

Table I

Type of Δ		h^0	h^1	h^2
I		6	0	0
		4	0	0
		3	$n-3$	0
II		2	0	0
II$\frac{1}{2}$		1	$n-4$	0
III	the other case	0	at least $2(n-4)-\sum_{j=1}^{s}(\nu_j-2)$	see (3)

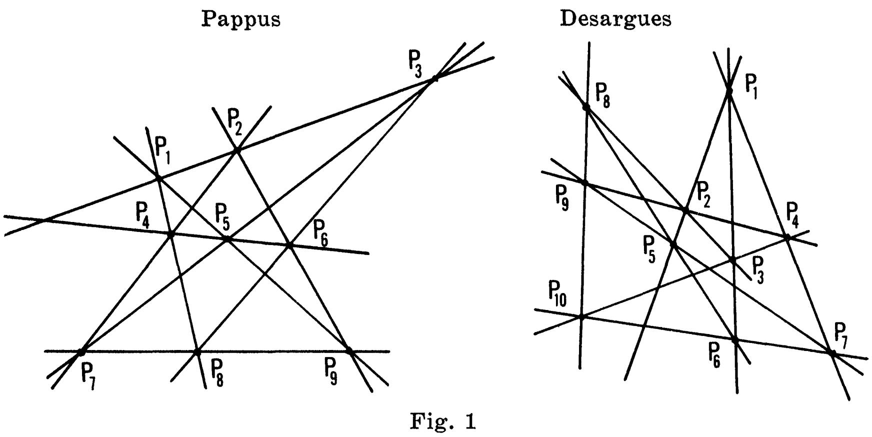


Fig. 1

structed from the complements of lines Δ in $\boldsymbol{P}^2$.

Claim 1. *We may assume that each D_i has $D_i^2<-1$.*

Proof. To show this, let $D=D'+D''$ be a decomposition such that D' is a sum of components D'_i of D with $D_i'^2<-1$ and D'' is a sum of components D''_i of D with $D_i''^2\geqq -1$. From the exact sequence (see [2])

$$0 \longrightarrow \Theta_X(\log D) \longrightarrow \Theta_X(\log D') \longrightarrow N^0_{D''/X} \longrightarrow 0,$$

we have

$$\cdots \longrightarrow H^1(X, \Theta_X(\log D)) \longrightarrow H^1(X, \Theta_X(\log D')) \longrightarrow H^1(N^0_{D''/X}) \longrightarrow \cdots,$$

where $H^1(N^0_{D''/X})=0$ by assumption.

Now we consider the versal family of $(X \backslash D', X, D')$, denoted by $(\mathcal{X}' \backslash \mathcal{D}', \mathcal{X}', \mathcal{D}', B', \pi')$. Let $\psi: \hat{B} \to B'$ be an induced mapping by forgetting the component D'' of D. From the above exact sequence, the tangent mapping

$$d\psi: T_0\hat{B} \longrightarrow T_{0'}B'$$

is surjective, where $T_0\hat{B}$ denotes the Zariski tangent space of $\hat{B}$ at 0. Hence $\psi: \hat{B} \to B'$ is smooth and therefore it is sufficient to show that $(X \backslash D', X, D')$ is unobstructed. Q.E.D.

Next we consider abstract deformations of X. Let $\bar{\mathcal{X}} \to \bar{B}$ be the versal family of deformations of X. Since $H^2(X, \Theta_X)=0$, $\bar{B}$ is non-singular and has the same dimension as $H^1(X, \Theta_X)$.

We may assume that the higher multiple points, $P_1, \cdots, P_s$ lie in general position with $s \geqq 4$ (see [3]). The family $\bar{\mathcal{X}} \to \bar{B}$ is given by moving arrangements of $P_1, \cdots, P_s$ being four general points fixed.

Now we consider each non-singular triple $(X \backslash D_i, X, D_i)$, forgetting other components, for $i=1, \cdots, n$. We have the versal family of logarithmic deformations of each non-singular triple $(X \backslash D_i, X, D_i)$, denoted by $(\mathcal{X}_i \backslash \mathcal{D}_i, \mathcal{X}_i, \mathcal{D}_i, B_i, \pi_i)$. Since $H^2(X, \Theta_X(\log D_i))=0$, each B_i is non-singular. By versality of $\bar{B}$ for an ambient space X, there exists an induced mapping $\psi_i: B_i \to \bar{B}$. From

$$0 \longrightarrow \Theta_X(\log D_i) \longrightarrow \Theta_X \longrightarrow N_{D_i/X} \longrightarrow 0,$$

we have

$$\cdots \longrightarrow H^0(N_{D_i/X}) \longrightarrow H^1(X, \Theta_X(\log D_i)) \longrightarrow H^1(X, \Theta_X) \longrightarrow \cdots.$$

Since $D_i^2 < -1$ and $H^0(N_{D_i/X})=0$, the tangent mapping

$$d\psi_i: T_0B_i = H^1(\Theta_X(\log D_i)) \longrightarrow T_0\bar{B} = H^1(\Theta_X)$$

is injective, hence ψ_i is an embedding.

The point is that each B_i is defined by the linear equations of parameters of $\bar{B}$, which is given by the collinear conditions of the points which D_i pass through. Let $B := B_1 \times_{\bar{B}} \times B_2 \times_{\bar{B}} \cdots \times_{\bar{B}} B_n$ and $\sigma: B \to \bar{B}$ the natural projection. Then B is a non-singular subspace of $\bar{B}$. Let $\mathcal{X}$ be $\sigma^*\bar{\mathcal{X}}$ and $\mathcal{D}$ be $\sigma_1^*\mathcal{D}_1 + \cdots + \sigma_n^*\mathcal{D}_n$, where $\sigma_i: B \to B_i$ are natural projections. Thus we can construct a family of logarithmic deformations $(\mathcal{X} \backslash \mathcal{D}, \mathcal{X}, \mathcal{D}, B, \pi)$ of $(X \backslash D, X, D)$.

Claim 2. *$(\mathcal{X} \backslash \mathcal{D}, \mathcal{X}, \mathcal{D}, B, \pi)$ is a versal family of logarithmic deformations of $(X \backslash D, X, D)$.*

Proof. We may assume $n=2$. The completeness follows immediately from the universality of a fiber product. It suffices to show that $T_0B = H^1(X, \Theta_X(\log D))$.

From the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta_X(\log D) & \longrightarrow & \Theta_X(\log D_1) & \longrightarrow & N_{D_2/X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_X(\log D_2) & \longrightarrow & \Theta_X & \longrightarrow & N_{D_2/X} & \longrightarrow & 0, \end{array}$$

we see, by diagram chasing,

$$H^1(X, \Theta_X(\log D)) \cong H^1(X, \Theta_X(\log D_1)) \times_{H^1(X,\Theta_X)} H^1(X, \Theta_X(\log D_2)).$$

Hence,

$$T_0B = T_0B_1 \times_{T_0B} T_0B_2 = H^1(X, \Theta_X(\log D)). \qquad \text{Q.E.D.}$$

This completes the proof.

§3. Examples with $H^2(X, \Theta(\log D)) \neq 0$. If Δ corresponds to Pappus's configuration, we see $h^0=0$, $h^1=2$ and $h^2=1$.

If Δ corresponds to Desargues's configuration, we see $h^0=0$, $h^1=3$ and $h^2=1$.

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References

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[3] H. Maeda: Deformations of complements of lines in P^2 (1981) (preprint).