15. Branching of Singularities for Degenerate Hyperbolic Operator and Stokes Phenomena. IV

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1983)

1. This note is a continuation of our previous notes [2]–[4]. The aims of this note are to generalize the results of Alinhac [1], Hanges [6] and Taniguchi-Tozaki [8] concerning the sufficient condition of branching of singularities for more general second order differential equations with higher order degeneracy and to show the computability of our condition. The details and further discussions will appear in [5].

2. **Review of [4] and results.** Let $t \in [-T, T]$, $x=(x_1, \cdots, x_n) \in R^n$, $D_t=\partial/(\sqrt{-1}\,\partial x_j)$ $(1 \leq j \leq n)$ and $P=P(t, X, D_t, D_x)$ be a second order linear partial differential operator of the form:

$$P=\sum_{j=0}^{2}\sum_{i=0}^{2-j} P_{i,j}(t, X, D_x)D_t^{2-j-i}$$

where each $P_{i,j}(t, x, \xi)$ is a homogeneous polynomial of degree i with respect to $\xi=(\xi_1, \cdots, \xi_n) \in R^n$. For simplicity, we assume all the coefficients have bounded derivatives of any order on $[-T, T] \times R^n_x$.

We assume the following conditions (A.1)–(A.3) for P which are invariant under a change of x variable.

(A.1) $P_2(t, x, \tau, \xi)$ is smoothly factorizable as follows:

$$P_2(t, x, \tau, \xi)=\prod_{j=0}^{2} (\tau - t^l \lambda_j(t, x, \xi))$$

where $l \in N$ and $\lambda_j(t, x, \xi) \in C^\infty([-T, T] \times R^n \times (R^n_\xi - \{0\}))$ $(j=1, 2)$ are real valued.

(A.2) There exists a constant $c>0$ such that

$$|\lambda_1(t, x, \xi) - \lambda_2(t, x, \xi)| \geq c|\xi| \qquad \text{for any } (t, x, \xi).$$

(A.3) Each $P_{i,j}(t, x, \xi)$ $(il \geq j,\ 2-j-i \geq 0)$ has the property:

$$P_{i,j}(t, x, \xi)=t^{il-j}\tilde{P}_{i,j}(t, x, \xi)$$

where $P_{i,j}(t, x, \xi)$ is a homogeneous polynomial of degree i in ξ and its coefficients have bounded derivatives of any order on $[-T, T] \times R^n_x$.

Now, set $L_0=D_t^2+t^l P_{1,0}(0, x, \xi)D_t + t^{l-1}P_{1,1}(0, x, \xi)+t^{2l}P_{2,0}(0, x, \xi)$ and denote the central connection coefficients associated to L_0 by $T_\pm^{(i,j)}(x, \xi)$ $(i, j=1, 2)$ (see [4] for the definition). Also, denote the (i, j)-cofactor of the matrix $\left(T_\pm^{(i,j)}(x, \xi)\,;\ \begin{smallmatrix} i \downarrow 1, 2 \\ j \to 1, 2 \end{smallmatrix}\right)$ by $T_\pm^{(i,j)}(x, \xi)$. Moreover, we denote

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the homogeneous symplectic transformation associated to P by $T_j(t, s)$ $(j=1, 2)$ (see [4] for the definition).

Then the following is the paraphrase of one of the result announced in [4] for the case operators are second order.

Theorem 1. *Let $i=0$ or 1 and assume the wave front set $WF(u_h)$ $(h=0, 1)$ of $u_h \in \mathcal{E}'(R^n)$ $(h=0, 1)$ satisfy the property: $\bigcup_{h\neq i} WF(u_h)=\varphi$, $WF(u_i)=\{(y^0, \rho\eta^0); \rho>0\}$. Let $u(t, s, x)$ be the solution of the Cauchy problem: $Pu=0$, $D_t^h u|_{t=s}=u_h$ $(h=0, 1)$. Suppose the following condition (♯)′ hold for a particular pair (ν_0, μ_0).*

$$(\sharp)' \qquad \sum_{j=1}^{2} T_{+}^{(j,\nu_0)}(y^0, \eta^0) T_{-}^{(j,\mu_0)}(y^0, \eta^0) \neq 0.$$

Then, there exists $T_0>0$ such that $T_{\nu_0}(t, 0) \circ T_{\mu_0}(0, s)(y^0, \eta^0) \in WF(u(t, s))$ for any $-T_0 \leq s < 0 < t \leq T_0$.

Next theorem shows the computability of the central connection coefficients.

Theorem 2. *Let $\alpha_i=\alpha_i(x, \xi)=\sqrt{-1}\,\lambda_i(0, x, \xi/|\xi|)$, $K_i=K_i(x, \xi)$ $=-H_i(x, \xi)/((l+1)G_i(x, \xi))$, $\gamma_i=\gamma_i(x, \xi)=K_i-1$ $(i=1, 2)$ and $\delta_2=(l+1)^{-1}$ where*

$$G_i(x, \xi)=\sum_{j=0}^{1}(2-j)\lambda_i(0, x, \xi)^{1-j} P_{j,0}(0, x, \xi),$$

$$H_i(x, \xi)=l\lambda_i(0, x, \xi)P_{j,0}(0, x, \xi)+\sqrt{-1}\sum_{j=0}^{1}\lambda_i(0, x, \xi)^{1-j}P_{j,1}(0, x, \xi).$$

Assume $K_i, K_i+\delta_2 \notin Z$ $(i=1, 2)$.

(1.i) *When $\lambda_1(0, x, \xi)>\lambda_2(0, x, \xi)$,*

$$\begin{cases} T_{+}^{(1,1)}=(l+1)^{K_1}\Gamma(\gamma_1+\delta_2+1)(\alpha_2-\alpha_1)^{\gamma_2+\delta_2}(1-\exp[-\pi\sqrt{-1}(\gamma_2+\delta_2)] \\ \qquad \cdot\sin\pi(\gamma_2+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2)), \\ T_{+}^{(1,2)}=(l+1)^{K_2}\Gamma(\gamma_2+\delta_2+1)(\alpha_1-\alpha_2)^{\gamma_1+\delta_2}\exp[-\pi\sqrt{-1}(\gamma_1+\delta_2)] \\ \qquad \cdot\sin\pi(\gamma_2+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2), \\ T_{+}^{(2,1)}=\sqrt{-1}\,|\xi|^{-\delta_2}(l+1)^{K_1+\delta_2}\Gamma(\gamma_1+1)(\alpha_2-\alpha_1)^{\gamma_2}(1-\exp[-\pi\sqrt{-1}\,\gamma_1] \\ \qquad \cdot\sin\pi\gamma_2/\sin\pi(\gamma_1+\gamma_2)), \\ T_{+}^{(2,2)}=\sqrt{-1}\,|\xi|^{\delta_2}(l+1)^{K_2+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_2+1)(\alpha_1-\alpha_2)^{\gamma_1}\exp[-\pi\sqrt{-1}\,\gamma_1] \\ \qquad \cdot\sin\pi\gamma_2/\sin\pi(\gamma_1+\gamma_2). \end{cases}$$

(1.ii) *When $\lambda_2(0, x, \xi)>\lambda_1(0, x, \xi)$,*

$$\begin{cases} T_{+}^{(1,1)}=(l+1)^{K_1}\Gamma(\delta_2)^{-1}\Gamma(\gamma_1+\delta_2+1)(\alpha_2-\alpha_1)^{\gamma_2+\delta_2}\exp[-\pi\sqrt{-1}(\gamma_2+\delta_2)] \\ \qquad \cdot\sin\pi(\gamma_1+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2), \\ T_{+}^{(1,2)}=(l+1)^{K_2}\Gamma(\delta_2)^{-1}\Gamma(\gamma_2+\delta_2+1)(\alpha_1-\alpha_2)^{\gamma_1+\delta_2}(1-\exp[-\pi\sqrt{-1}(\gamma_2+\delta_2)] \\ \qquad \cdot\sin\pi(\gamma_1+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2)), \\ T_{+}^{(2,1)}=\sqrt{-1}\,|\xi|^{-\delta_2}(l+1)^{K_1+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_1+1)(\alpha_2-\alpha_1)^{\gamma_2}\exp[-\pi\sqrt{-1}\,\gamma_2] \\ \qquad \cdot\sin\pi\gamma_1/\sin\pi(\gamma_1+\gamma_2), \\ T_{+}^{(2,2)}=\sqrt{-1}\,|\xi|^{-\delta_2}(l+1)^{K_2+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_2+1)(\alpha_1-\alpha_2)^{\gamma_1} \\ \qquad \cdot(1-\exp[-\pi\sqrt{-1}\,\gamma_2]\sin\pi\gamma_1/\sin\pi(\gamma_1+\gamma_2)). \end{cases}$$

(2) *In the case l is odd, we have*

$$\begin{cases} T_{-}^{(1,j)}=(-1)^{\mu_j(x,\xi)}T_{+}^{(1,j)}(x,\xi) \\ T_{-}^{(2,j)}=(-1)^{\mu_j(x,\xi)+1}T_{+}^{(2,j)}(x,\xi) \end{cases} \quad (j=1,2).$$

(3) *In the case l is even, we have the following two cases.*

(3.i) *When* $\lambda_1(0,x,\xi)>\lambda_2(0,x,\xi)$,

$$\begin{cases} T_{-}^{(1,1)}=(-1)^{\mu_1}(l+1)^{K_1}\Gamma(\delta_2)^{-1}\Gamma(\gamma_1+\delta_2+1)(\alpha_1-\alpha_2)^{\gamma_2+\delta_2} \\ \quad\cdot\exp[-\pi\sqrt{-1}(\gamma_2+\delta_2)]\sin\pi(\gamma_1+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2), \\ T_{-}^{(1,2)}=(-1)^{\mu_2}(l+1)^{K_2}\Gamma(\delta_2)^{-1}\Gamma(\gamma_2+\delta_2+1)(\alpha_2-\alpha_1)^{\gamma_1+\delta_2} \\ \quad\cdot(1-\exp[-\pi\sqrt{-1}(\gamma_2+\delta_2)]\sin\pi(\gamma_1+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2)), \\ T_{-}^{(2,1)}=\sqrt{-1}|\xi|^{-\delta_2}(-1)^{\mu_1+1}(l+1)^{K_1+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_1+1)(\alpha_1-\alpha_2)^{\gamma_2} \\ \quad\cdot\exp[-\pi\sqrt{-1}\gamma_2]\sin\pi\gamma_1/\sin\pi(\gamma_1+\gamma_2), \\ T_{-}^{(2,2)}=\sqrt{-1}|\xi|^{-\delta_2}(-1)^{\mu_2+1}(l+1)^{K_2+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_2+1)(\alpha_2-\alpha_1)^{\gamma_1} \\ \quad\cdot(1-\exp[-\pi\sqrt{-1}\gamma_2]\sin\pi\gamma_1/\sin\pi(\gamma_1+\gamma_2)). \end{cases}$$

(3.ii) *When* $\lambda_2(0,x,\xi)>\lambda_1(0,x,\xi)$,

$$\begin{cases} T_{-}^{(1,1)}=(-1)^{\mu_1}(l+1)^{K_1}\Gamma(\gamma_2)^{-1}\Gamma(\gamma_1+\delta_2+1)(\alpha_1-\alpha_2)^{\gamma_2} \\ \quad\cdot(1-\exp[-\pi\sqrt{-1}(\gamma_1+\delta_2)]\sin\pi(\gamma_2+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2)), \\ T_{-}^{(1,2)}=(-1)^{\mu_2}(l+1)^{K_2}\Gamma(\delta_2)^{-1}\Gamma(\gamma_2+\delta_2+1)^{\gamma_1+\delta_2}\exp[-\pi\sqrt{-1}(\gamma_1+\delta_2)] \\ \quad\cdot\sin\pi(\gamma_2+\delta_2)/\sin\pi(\gamma_1+\gamma_2+2\delta_2), \\ T_{-}^{(2,1)}=\sqrt{-1}|\xi|^{-\delta_2}(-1)^{\mu_1+1}(l+1)^{K_1+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_1+1)(\alpha_1-\alpha_2)^{\gamma_2} \\ \quad\cdot(1-\exp[-\pi\sqrt{-1}\gamma_1]\sin\pi\gamma_2/\sin\pi(\gamma_1+\gamma_2)), \\ T_{-}^{(2,2)}=\sqrt{-1}|\xi|^{-\delta_2}(-1)^{\mu_2+1}(l+1)^{K_2+\delta_2}\Gamma(-\delta_2)^{-1}\Gamma(\gamma_2+1)(\alpha_2-\alpha_1)^{\gamma_1} \\ \quad\cdot\exp[-\pi\sqrt{-1}\gamma_1]\sin\pi\gamma_2/\sin\pi(\gamma_1+\gamma_2). \end{cases}$$

Here $\Gamma(\sigma)$ denotes the gamma function.

Remark. (1) Taking account of the fact that the change of variable $z=(l+1)^{-1}t^{l+1}|\xi|$ reduces L_0 to confluent type operator, our proof relies heavily on the global theory of confluent type operators (see [7]). This is the reason why we have to restrict the order of our operator.

(2) Restricting to the special type of operator $P=D_t^2-t^{2l}D_x^2+\sqrt{-1}at^{l-1}D_x$, where a is constant and $x\in\boldsymbol{R}^1$, we can deduce the results of Hanges [6] and Taniguchi-Tozaki [8] for this operator by combining the present result with the previous result of [2].

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