133. *Semivariation and Operator Semivariation of Hilbert Space Valued Measures*

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§1. Introduction. Let $\mathfrak{H}$ and $\mathfrak{K}$ be a pair of Hilbert spaces. $B(\mathfrak{H})$ denotes the Banach space of all bounded linear operators on $\mathfrak{H}$ with the identity 1 and the uniform norm $\|\cdot\|$, and $T(\mathfrak{H})$ denotes the set of all trace class operators on $\mathfrak{H}$ with the trace $\mathrm{Tr}\,(\cdot)$ and the trace norm $\|\cdot\|_\tau$. Let $X=S(\mathfrak{H},\mathfrak{K})$ be the set of all Hilbert-Schmidt class operators from $\mathfrak{H}$ into $\mathfrak{K}$. For $x, y\in X$ define $[x,y]=x^*y\in T(\mathfrak{H})$, $\langle x,y\rangle_X=\mathrm{Tr}\,[x,y]$ and $\|x\|_X=\langle x,x\rangle_X^{1/2}$. Then X becomes a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_X$.

Let $(\Omega,\mathfrak{F})$ be a measurable space. We consider X-valued measures defined on $\mathfrak{F}$. Denote by $ca(\Omega\,;X)$ the set of all X-valued bounded and countably additive, in the norm $\|\cdot\|_X$, measures on $\mathfrak{F}$. The *operator semivariation* of $\xi\in ca(\Omega\,;X)$ is the function $\|\xi\|_o(\cdot)$ whose value on a set $A\in\mathfrak{F}$ is given by

$$\|\xi\|_o(A)=\sup\left\|\sum_{k=1}^n \xi(A_k)a_k\right\|_X \tag{1.1}$$

where the supremum is taken for all finite measurable partitions $\{A_1,\cdots,A_n\}$ of A and for all finite subsets $\{a_1,\cdots,a_n\}\subset B(\mathfrak{H})$ with $\|a_k\|\leqslant 1$, $1\leqslant k\leqslant n$ (cf. Kakihara [2, Definition 3.1 and §5]). The *semivariation* of ξ is the function $\|\xi\|(\cdot)$ whose value on a set $A\in\mathfrak{F}$ is given in (1.1) by replacing $a_k\in B(\mathfrak{H})$ with $\lambda_k\in C$ (the complex number field) such that $|\lambda_k|\leqslant 1$, $1\leqslant k\leqslant n$ (cf. Diestel and Uhl [1, pp. 2–4]). Then we have $\|\xi(A)\|_X\leqslant\|\xi\|(A)\leqslant\|\xi\|_o(A)$, $A\in\mathfrak{F}$. In §2, we shall obtain the characterization of those measures $\xi\in ca(\Omega\,;X)$ for which the following condition is satisfied:

$$\|\xi\|_o(A)=\|\xi(A)\|_X,\qquad A\in\mathfrak{F}. \tag{1.2}$$

In §3, we consider the set $ca(\Omega\,;\mathfrak{H})$ of all $\mathfrak{H}$-valued bounded and countably additive measures on $\mathfrak{F}$. The *operator semivariation* of $\xi\in ca(\Omega\,;\mathfrak{H})$ is the function $\|\xi\|_o(\cdot)$ whose value on a set $A\in\mathfrak{F}$ is given by

$$\|\xi\|_o(A)=\sup\left\|\sum_{k=1}^n a_k\xi(A_k)\right\| \tag{1.3}$$

where the supremum is taken as in the case of (1.1) and $\|\cdot\|$ is the norm of $\mathfrak{H}$. If we identify $\mathfrak{H}$ with $S(\mathfrak{H},C)$, we see that the operator semivariations defined by (1.1) and (1.3) are identical for every $\xi \in ca(\Omega ; \mathfrak{H})$. We also consider the following conditions for $\xi \in ca(\Omega ; \mathfrak{H})$:

(1.4) $$\|\xi\|(A)=\|\xi(A)\|, \qquad A \in \mathfrak{F};$$

(1.5) $$\|\xi\|_0(A)=\|\xi(A)\|, \qquad A \in \mathfrak{F}.$$

$\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathfrak{H}$ and $U(\mathfrak{H})$ the set of all unitary operators on $\mathfrak{H}$. Notations introduced in this section will be used throughout this paper.

§ 2. Hilbert-Schmidt class operator valued measures. Our main theorem characterizing the condition (1.2) is stated as follows.

Theorem 2.1. *An X-valued measure $\xi \in ca(\Omega ; X)$ satisfies that $\|\xi\|_0(A)=\|\xi(A)\|_X$ for every $A \in \mathfrak{F}$ if and only if it satisfies that $[\xi(A), \xi(B)] \geqslant 0$ for every $A, B \in \mathfrak{F}$.*

Proof. Assume the condition (1.2) for ξ. It suffices to show that $[\xi(A), \xi(B)] \geqslant 0$ for every disjoint $A, B \in \mathfrak{F}$. Let $A, B \in \mathfrak{F}$ be disjoint. Then we have

$$\|\xi(A)+\xi(B)\|_X=\|\xi(A \cup B)\|_X=\|\xi\|_0(A \cup B) \geqslant \|\xi(A)+\xi(B)u\|_X$$

for every $u \in U(\mathfrak{H})$. Hence

$$\begin{aligned}\|\xi(A)+\xi(B)\|_X^2-\|\xi(A)+\xi(B)u\|_X^2 &=2 \operatorname{Re}\{\langle \xi(A), \xi(B)\rangle_X-\langle \xi(A), \xi(B)u\rangle_X\} \\ &=2 \operatorname{Re}\{\operatorname{Tr}(1-u)\xi(A)^*\xi(B)\} \geqslant 0\end{aligned}$$

where $\operatorname{Re}\{\cdots\}$ means the real part. It follows from Schatten [3, p. 43, Theorem 6] that $\xi(A)^*\xi(B)=[\xi(A), \xi(B)] \geqslant 0$.

Conversely suppose that $[\xi(A), \xi(B)] \geqslant 0$ for every $A, B \in \mathfrak{F}$. Take $A \in \mathfrak{F}$ and let $\{A_1, \cdots, A_n\} \subset \mathfrak{F}$ be a finite partition of A and $\{a_1, \cdots, a_n\} \subset B(\mathfrak{H})$ be such that $\|a_k\| \leqslant 1$, $1 \leqslant k \leqslant n$. Then, putting $x_k=\xi(A_k)$, we have

$$\begin{aligned}&\|\xi(A)\|_X^2-\left\|\sum_{k=1}^n \xi(A_k)a_k\right\|_X^2 \\ &=\sum_{k=1}^n (\|x_k\|_X^2-\|x_k a_k\|_X^2)+2 \operatorname{Re}\left\{\sum_{j>k} (\langle x_j, x_k\rangle_X-\langle x_j a_j, x_k a_k\rangle_X)\right\} \\ &\geqslant \sum_{k=1}^n (1-\|a_k\|^2)\|x_k\|_X^2+2 \operatorname{Re}\left\{\sum_{j>k} (\|x_j^* x_k\|_\tau-\operatorname{Tr}(a_j^* x_j^* x_k a_k))\right\} \\ &\geqslant 2 \sum_{j=k} (1-\|a_k a_j^*\|)\|x_j^* x_k\|_\tau \geqslant 0\end{aligned}$$

since $x_j^* x_k \geqslant 0$ and $\|a_k\| \leqslant 1, 1 \leqslant j, k \leqslant n$. Hence we have $\|\xi\|_0(A) \leqslant \|\xi(A)\|_X$ and, therefore, $\|\xi\|_0(A)=\|\xi(A)\|_X$.

Remark 2.2. (1) If $\xi \in ca(\Omega ; X)$ is *orthogonally scattered*, i.e., $[\xi(A), \xi(B)]=0$ for every disjoint $A, B \in \mathfrak{F}$, then it necessarily satisfies the condition (1.2) by Theorem 2.1. This fact was proved in [2, Proposition 5.5]. (2) It follows from the above proof that, for $x, y \in X$, $[x, y] \geqslant 0$ iff $\|x+y\|_X \geqslant \|x+yu\|_X$, $u \in U(\mathfrak{H})$. Hence, $[x, y]=0$ iff $\|x \pm y\|_X \geqslant \|x+yu\|_X$, $u \in U(\mathfrak{H})$.

§ 3. Hilbert space valued measures. According to Theorem 2.1, we can characterize the conditions (1.4) and (1.5) for an $\mathfrak{H}$-valued measure $\xi \in ca(\Omega ; \mathfrak{H})$. Interchanging $\mathfrak{H}$ with $\mathfrak{R}$ and letting $\mathfrak{R}=C$ in Theorem 2.1, we obtain the following corollary.

Corollary 3.1. *An $\mathfrak{H}$-valued measure $\xi \in ca(\Omega ; \mathfrak{H})$ satisfies that $\|\xi\|(A)=\|\xi(A)\|$ for every $A \in \mathfrak{F}$ if and only if it satisfies that $\langle \xi(A), \xi(B)\rangle \geqslant 0$ for every $A, B \in \mathfrak{F}$.*

It follows from Remark 2.2 that, for $\phi, \psi \in \mathfrak{H}$, $\langle \phi, \psi\rangle \geqslant 0$ iff $\|\phi+\psi\| \geqslant \|\phi+\alpha\psi\|$, $\alpha \in C$ with $|\alpha|=1$ and, hence, $\langle \phi, \psi\rangle=0$ iff $\|\phi\pm\psi\| \geqslant \|\phi+\alpha\psi\|$, $\alpha \in C$ with $|\alpha|=1$.

The following shows that an $\mathfrak{H}$-valued measure ξ satisfying the condition (1.5) is essentially a scalar valued measure.

Proposition 3.2. *An $\mathfrak{H}$-valued measure $\xi \in ca(\Omega ; \mathfrak{H})$ satisfies that $\|\xi\|_0(A)=\|\xi(A)\|$ for every $A \in \mathfrak{F}$ if and only if there exist a finite nonnegative measure μ on $\mathfrak{F}$ and a vector $\phi \in \mathfrak{H}$ such that $\xi(\cdot)=\mu(\cdot)\phi$.*

Proof. If we identify $\mathfrak{H}$ with $S(\mathfrak{H}, C)$, then $[\phi, \psi]=\psi\otimes\bar{\phi}$ for $\phi, \psi \in \mathfrak{H}$ where the tensor product $\otimes$ is in the sense of [3]. Note that $[\phi, \psi]\geqslant 0$ iff $\langle \phi, \psi\rangle=\mathrm{Tr}\,(\psi\otimes\bar{\phi})=\|\psi\otimes\bar{\phi}\|_\tau=\|\psi\|\cdot\|\phi\|$ iff there is some $\lambda\geqslant 0$ such that $\phi=\lambda\psi$ or $\psi=\lambda\phi$. Since the "if" part is easy to verify, we prove the "only if" part. Without loss of generality we may assume that there is some $B \in \mathfrak{F}$ such that $\xi(B)\neq 0$. Put $\phi=\xi(B)$. By Theorem 2.1 we have that $[\xi(A), \xi(B)]=\phi\otimes\overline{\xi(A)}\geqslant 0$ for each $A \in \mathfrak{F}$. Hence there exists some $\mu(A)\geqslant 0$ such that $\xi(A)=\mu(A)\phi$ for each $A \in \mathfrak{F}$. It is immediate that μ is a finite nonnegative measure on $\mathfrak{F}$. Therefore the proof is complete.

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