130. *Fundamental Theorems in Global Knot Theory. I*

By Itiro TAMURA
Department of Mathematics, Faculty of Science, University of Tokyo

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1. Introduction. A *knot* K in an m-dimensional smooth manifold M^m is a submanifold of M^m diffeomorphic to the $(m-2)$-sphere S^{m-2}. The knots in manifolds are the object of global knot theory which is a generalization of the knot theory in spheres. Each manifold has its own knot theory (see [6, Theorem 1]).

A knot K in M^m is said to be *unknotted* if K bounds an $(m-1)$-disk smoothly imbedded in M^m. A knot K in M^m is said to be *local* if there exists an m-disk D^m smoothly imbedded in M^m such that $D^m \supset K$.

Criterion theorems of unknottedness and localness for knots in manifolds are fundamental in global knot theory as the unknotting theorem of Papakyriakopoulos and that of Levine are fundamental in the classical knot theory and the higher dimensional knot theory ([3], [2]).

In this note we announce the unknotting and localness theorems for knots in highly connected smooth manifolds by means of knot modules (Theorems 3 and 4), generalizing the unknotting theorem by Levine. As an application the localness and the unknottedness of genus 1 knots in $S^n \times S^{n+1}$ will be determined by computing their knot modules ([6, Theorem 1]).

2. Seifert surfaces. Let K be a knot in a connected m-dimensional smooth manifold M^m $(m \geqq 3)$. A compact connected $(m-1)$-dimensional submanifold V of M^m such that $\partial V = K$ is called a *Seifert surface* for K if V is transversally orientable.

Proposition 1. *Let M^m be a connected closed m-dimensional smooth manifold such that $m \geqq 4$ and $H^2(M^m; \boldsymbol{Z}) = 0$, and let K be a knot in M^m. Then there exists a Seifert surface for K.*

A knot K in M^m is said to be *r-simple* if the homotopy groups $\pi_i(X)$ of the complement $X = M^m - K$ are as follows:

$$\pi_1(X) \cong \boldsymbol{Z}, \quad \pi_i(X) = 0 \qquad \text{for } 2 \leqq i \leqq r.$$

Proposition 2. *Let M^m be a q-connected closed m-dimensional smooth manifold and let K be a knot in M^m, where $m \geqq 6$, $2 \leqq q \leqq [m/2] - 1$.*

(a) *If K is k-simple for* $1 \leqq k \leqq q$ (resp. $1 \leqq k \leqq q-1$) *in case m is odd* (resp. m is even), *then there exists a Seifert surface for K which is k-connected.*

(b) *Conversely if there exists a k'-connected Seifert surface for K such that $1 \leqq k' \leqq q$, then the knot K is k'-simple.*

The proof is similar to that of the case of higher dimensional knots in spheres (Levine [2, Theorem 2]).

3. Fibred knots in highly connected manifolds. A knot K in M^m is said to be *fibred* if the normal bundle of K is trivial and there exists a smooth fibration $\bar{p}: X \to S^1$ such that the restriction $\bar{p} | \partial N(K): \partial N(K) \to S^1$ is the projection onto a fibre of the fibration $\partial N(K) \to K$, where $N(K)$ is a tubular neighborhood of K.

In case M^m is an $(n-1)$-connected closed $(2n+1)$-dimensional (resp. an $(n-1)$-connected closed $2n$-dimensional) smooth manifold, an $(n-1)$-simple (resp. $(n-2)$-simple) knot K in M^m is said to be *simple,* where $n \geqq 2$ (resp. $n \geqq 3$).

The following theorem was proved in order to show the existence of foliations of codimension one ([4, Theorem 8]).

Theorem 1. *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n \geqq 3)$. Then there exists a knot K in M^{2n+1} which is simple and fibred.*

As a corollary of this theorem we have the following theorem.

Theorem 2. *There exists a simple knot K in $S^n \times S^{n+1}$ $(n \geqq 3)$ such that K is fibred, thus not local, and inessential.*

The inessentiality is necessary for a knot to be local. But this theorem shows that it is not sufficient. This answers Problem 1 in [5].

4. Knot modules. Let K be a knot in a 2-connected compact m-dimensional smooth manifold M^m. The complement X is the most important invariant of K. Let $p: \tilde{X} \to X$ be the maximal abelian covering corresponding to the kernel of the surjection $\pi_1(X) \to H_1(X) \cong \boldsymbol{Z}$ and let t denote a multiplicative generator of the covering transformation group $H_1(X)$. The homomorphism $H_q(\tilde{X}) \to H_q(\tilde{X})$ induced by $t: \tilde{X} \to \tilde{X}$ is also denoted by the same notation t, and we denote $t(\alpha)$ by $t\alpha$ for $\alpha \in H_q(\tilde{X})$. The integral group ring Λ of $H_1(X)$ is the ring $\boldsymbol{Z}[t, t^{-1}]$ of Laurent polynomials in t.

The homology modules $A_q(K; M^m) = H_q(\tilde{X})$ $(q \geqq 1)$ are called the *knot modules* of K. $A_q(K; M^m)$ is a finitely generated module over Λ. An intersection pairing I of $H_q(\tilde{X})$ is defined by fixing an orientation of $\tilde{X}$ (Blanchfield [1]):

$$I: A_q(K; M^m) \otimes A_{m-q}(K; M^m) \longrightarrow \boldsymbol{Z}.$$

5. Localness theorem and Unknotting theorem. Theorem 3 (Localness theorem). (I) *Let M^{2n} be an $(n-1)$-connected closed $2n$-dimensional smooth manifold $(n \geqq 3)$ and let K be a simple knot in M^{2n}. Then K is local if and only if $A_n(K; M^{2n})$ contains a direct summand $\bar{A}$ satisfying the following conditions:*

(a) $\bar{A}$ *is a Λ-free module of rank b, where b is the n-th Betti number of M^{2n}.*

(b) *There exists a Λ-basis $\{\omega_1, \omega_2, \cdots, \omega_b\}$ for $\bar{A}$ such that*

$$(*) \qquad I(\omega_i, t^k\omega_j)=0 \qquad k\neq 0;\ i, j=1, 2, \cdots, b.$$

(II) *Let M^{2n+1} be an $(n-1)$-connected closed $(2n+1)$-dimensional smooth manifold $(n\geqq 3)$ such that $H_n(M^{2n+1})$ is torsion free, and let K be a simple knot in M^{2n+1}. Then K is local if and only if $A_n(K; M^{2n+1})$ and $A_{n+1}(K; M^{2n+1})$ contain direct summands $\bar{B}$ and $\bar{C}$ respectively which satisfy the following conditions:*

(a) *$\bar{B}$ and $\bar{C}$ are Λ-free modules of rank b, where b is the n-th and the $(n+1)$-th Betti number of M^{2n+1}.*

(b) *There exist a Λ-basis $\{\xi_1, \xi_2, \cdots, \xi_b\}$ for $\bar{B}$ and a Λ-basis $\{\zeta_1, \zeta_2, \cdots, \zeta_b\}$ for $\bar{C}$ such that*

$$(**) \qquad I(\xi_i, \zeta_j)=\delta_{ij}, \quad I(\xi_i, t^k\zeta_j)=0 \quad i, j=1, 2, \cdots, b;\ k\neq 0.$$

Theorem 4 (Unknotting theorem). (I) *Let M^{2n} be as in Theorem* 3, (I) *and let K be a* 1*-simple knot in M^{2n}. Then K is unknotted if and only if $A_q(K; M^{2n})$ $(2\leqq q\leqq n)$ have the following properties:*

(a) $A_q(K; M^{2n})=0 \qquad 2\leqq q\leqq n-1.$

(b) *$A_n(K; M^{2n})$ is a free Λ-module of rank b with a Λ-basis $\{\omega_1, \omega_2, \cdots, \omega_b\}$ such that they satisfy the condition* $(*)$ *in Theorem* 3, (I), *where b is the n-th Betti number of M^{2n}.*

(II) *Let M^{2n+1} be as in Theorem* 3, (II) *and let K be a* 1*-simple knot in M^{2n+1}. Then K is unknotted if and only if $A_q(K; M^{2n+1})$ $(2\leqq q\leqq n+1)$ have the following properties:*

(a) $A_q(K; M^{2n+1})=0 \qquad 2\leqq q\leqq n-1.$

(b) *$A_n(K; M^{2n+1})$ and $A_{n+1}(K; M^{2n+1})$ are free Λ-modules of rank b with Λ-basis $\{\xi_1, \xi_2, \cdots, \xi_b\}$ and $\{\zeta_1, \zeta_2, \cdots, \zeta_b\}$ respectively such that they satisfy the condition* $(**)$ *in Theorem* 3, (II), *where b is the n-th and the $(n+1)$-th Betti number of M^{2n+1}.*

Details and proofs will appear elsewhere.

References

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