123. A Counterexample to a Problem on Commuting Matrices

By Jiro SEKIGUCHI
Department of Mathematics, Tokyo Metropolitan University

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§1. **Formulation of the problem.** Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and put

$$C=\{(X, Y)\in\mathfrak{g}\times\mathfrak{g}\,;\,[X, Y]=0\}.$$

It is known (cf. [4]) that C is an irreducible algebraic variety. As an easy consequence of this result, we find that

(1) $\quad \dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)\geq \operatorname{rank}\mathfrak{g} \quad$ for any $(X, Y)\in C$.

Here $\mathfrak{g}_X$ and $\mathfrak{g}_Y$ denote the centralizers of X and Y, respectively. Then Prof. M. Kashiwara asked the author the following

Problem. *Let $X\in\mathfrak{g}$. Then does there exist a $Y\in\mathfrak{g}$ such that Y commutes with X and that* $\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)=\operatorname{rank}\mathfrak{g}$?

This problem connects with the study of the holonomic system of differential equations which governs an invariant eigendistribution on a real form of $\mathfrak{g}$. For the details, see [2, §6].

The purpose of this short note is to give a counterexample to this problem when $\mathfrak{g}$ is simple of type F_4 and $X\in\mathfrak{g}$ is a certain nilpotent element.

We note here some remarks on the problem.

(1) If $X\in\mathfrak{g}$ is regular, that is, $\dim\mathfrak{g}_X=\operatorname{rank}\mathfrak{g}$, it is known (cf. [3]) that $\mathfrak{g}_X$ is abelian and therefore $\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)=\operatorname{rank}\mathfrak{g}$ for any $Y\in\mathfrak{g}_X$.

(2) It is easy to reduce the problem to the case when X is a distinguished nilpotent element of $\mathfrak{g}$.

(3) Assume that $\mathfrak{g}$ is simple and the type of $\mathfrak{g}$ is one of $A_l, B_l, C_l, D_l, E_6, E_7, G_2$. Then for any distinguished nilpotent $X\in\mathfrak{g}$, there exists a $Y\in\mathfrak{g}$ such that Y commutes with X and $\dim\mathfrak{g}_X\cap\mathfrak{g}_Y=\operatorname{rank}\mathfrak{g}$. Namely, the problem is true in these cases. The details of this result will be published elsewhere.

(4) In the case when $\mathfrak{g}$ is simple of type E_8, the problem is rest open.

§2. **A counterexample to the problem.** Let $\mathfrak{g}$ be a simple Lie algebra of type F_4 and let X be a nilpotent element of $\mathfrak{g}$ whose weighted Dynkin diagram is $02\Rightarrow00$ (cf. [1]).

Claim. *For any $Y\in\mathfrak{g}_X$, we have*

$$\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)\geq 6.$$

Since rank $\mathfrak{g}=4$, this is a counterexample to the problem.

Proof of the claim. By Jacobson-Morozov lemma, there exist H, $Y\in\mathfrak{g}$ satisfying the commutation relation

$$[H,X]=2X,\quad [H,Y]=-2Y,\quad [X,Y]=H.$$

Then adH induces an endomorphism of $\mathfrak{g}_X$. It is known (cf. [1]) that $\dim\mathfrak{g}_X=12$. By direct calculation, we find that the eigenvalues of $adH|\mathfrak{g}_X$ are 2, 4, 6 and that if we put $V_k=\{Z\in\mathfrak{g}_X\,;\,[H,Z]=kZ\}$, then $\dim V_2=6$, $\dim V_4=4$, $\dim V_6=2$.

We are now going to prove the claim. Let Z be an arbitrary element of $\mathfrak{g}$ commuting with X. Put $Z=Z_2+Z_4+Z_6$, where $[H,Z_k]=kZ_k$ $(k=2,4,6)$. Then it is clear that $Z_k\in V_k$ $(k=2,4,6)$. We denote $\mathfrak{z}=\mathfrak{g}_X\cap\mathfrak{g}_Z$. Since Z_6 is contained in the center of $\mathfrak{z}$, it follows that $\mathfrak{z}=\mathfrak{g}_X\cap\mathfrak{g}_{Z_2+Z_4}$. Hence we may assume that $Z_6=0$ without loss of generality.

First assume that $Z_4=0$. Then $Z=Z_2\in V_2$. If Z is a constant multiple of X, we have nothing to prove. Hence we assume that $Z\notin CX$. Since $[Z,V_4]\subset V_6$ and since $\dim V_4=4$ and $\dim V_6=2$, there exist linearly independent elements $u_1,u_2\in V_4$ such that $[Z,u_i]=0$ $(i=1,2)$. Then u_1,u_2,X,Y and V_6 are contained in $\mathfrak{z}$ and this implies that $\dim\mathfrak{z}\geq 6$.

Next we consider the case when $Z_4\neq 0$. If $A\in V_4$, then $[Z,A]=0$ is equivalent to $[Z_2,A]=0$. Then by an argument similar to the above case, we find that there are linearly independent elements $u_1,u_2\in V_4$ commuting with Z. Hence also by an argument similar to the above, we conclude that $\dim\mathfrak{z}\geq 6$. Q.E.D.

References

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