119. Boundary Value Problems for Some Degenerate Elliptic Equations of Second Order

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1. Let Ω be a bounded domain in R^N with C^∞ boundary $\partial\Omega$ and $a^{ij}(x)=a^{ji}(x)$, $b^j(x)$ and $c(x)$ be real valued functions belonging to $C^\infty(\bar{\Omega})$. In this note we shall consider the regularity up to the boundary of the solution for the following boundary value problem:

$$\text{[P]}\quad Au=\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{j=1}^{N} b^j(x)\frac{\partial u}{\partial x_j}+c(x)u=f(x) \qquad \text{in } \Omega,$$

$$u|_{\partial\Omega}=0,$$

under the assumptions on A:

A1 $\quad a_2(x,\xi)=\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geqq 0 \qquad$ for $(x,\xi)\in\bar{\Omega}\times(R^N\backslash\{0\})$.

A2 $\quad c(x)<0$ and $c^*(x)=c(x)-\sum_{j=1}^{N}\frac{\partial b^j}{\partial x_j}(x)+\sum_{i,j=1}^{N}\frac{\partial^2 a^{ij}}{\partial x_i\partial x_j}(x)<0$ on $\bar{\Omega}$.

A3 $\quad \partial\Omega$ is non-characteristic for A.

A4 $\quad a_1^s(x,\xi)=\sum_{j=1}^{N}\left\{b^j(x)-\sum_{i=1}^{N}\frac{\partial a^{ij}}{\partial x_i}(x)\right\}\xi_j\neq 0$

for $(x,\xi)\in\sum=\{(x,\xi)\in\bar{\Omega}\times(R^N\backslash\{0\})\,|\,a_2(x,\xi)=0\}$.

Several existence, uniqueness and regularity theorems of the problem [P] were proved in Fichera [1], [2], Kohn-Nirenberg [4] and Oleinik [5], Oleinik-Radkevič [6]. In fact, it is known that there is a uniquely determined weak solution $u\in L^2(\Omega)$ of [P] with $f\in L^2(\Omega)$ if the conditions A1, A2 and A3 hold. Here $u\in L^2(\Omega)$ is called the weak solution of [P] with $f\in L^2(\Omega)$ if the identity

$$(1.1)\quad \int_\Omega u\overline{A^t v}\,dx=\int_\Omega f\bar{v}\,dx \qquad \text{holds for all } v\in C^\infty(\bar{\Omega}) \text{ with } v|_{\partial\Omega}=0,$$

where

$$A^t v=\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}-\sum_{j=1}^{N}\left\{b^j(x)-2\sum_{i=1}^{N}\frac{\partial a^{ij}}{\partial x_i}(x)\right\}\frac{\partial v}{\partial x_j}+c^*(x)v.$$

Concerning the local regularity of this weak solution, we can apply Theorem 5.9 in Hörmander [3] to the operator A if the conditions A1 and A4 hold (see also Radkevič [7]). That is, if u is the weak solution of [P] with $f\in H^k(\Omega)$,[1] then we have $u\in H^{k+1}(U)$ for any open set U such that $\bar{U}\subset\Omega$. This is the reason why we consider the boundary value problem [P].

1) $H^k(\Omega)$ denotes the Sobolev space on Ω for non-negative integer k.

Theorem 1. *Assume that the conditions* A1, A2, A3 *and* A4 *hold. Then the weak solution* $u \in L^2(\Omega)$ *of* [P] *with* $f \in H^k(\Omega)$ *belongs to* $H^{k+1}(\Omega)$. *In particular, if* $f \in C^\infty(\bar{\Omega})$, *then* $u \in C^\infty(\bar{\Omega})$.

Remark. H. Yamada [8] proposed a sufficient condition for the weak solution $u \in L^2(\Omega)$ of [P] with $f \in C^\infty(\bar{\Omega})$ to be in $C^\infty(\bar{\Omega})$. But his condition is more restrictive than ours, i.e., strong ellipticity on $\partial\Omega$ is assumed besides A1, A2 and A3.

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2. Let x_0 be any point on $\partial\Omega$. Since the boundary $\partial\Omega$ is non-characteristic for A, we can choose a neighborhood U of x_0 such that

(i) the boundary $\partial\Omega$ is described by the equation $\phi(x)=0$ in U, where $\nabla\phi\neq 0$ on $\partial\Omega$ and $\phi>0$ in Ω.

(ii) there are n independent C^∞ functions $\theta_1, \cdots, \theta_n$, where $n=N-1$, such that $\sum_{i,j=1}^N a^{ij}(x)(\partial\phi/\partial x_i)(\partial\theta_k/\partial x_j)=0$ in U and $\theta_k(x_0)=0$ for $k=1,2,\cdots,n$.

Thus there exist a neighborhood V of x_0 ($V\subset U$) and a diffeomorphism ϕ of the form $y=\phi(x)$, $x'_k=\theta_k(x)$ for $k=1,2,\cdots,n$ such that

(i) the image of $V\cap\Omega$ under Φ is $Q_{\delta_0}=(0,\delta_0)\times B_{\delta_0}$, where $B_{\delta_0}=\{x'\in R^n \mid |x'|<\delta_0\}$ with $\delta_0>0$, and the image of $V\cap\partial\Omega$ under Φ is $\{0\}\times B_{\delta_0}$.

(ii) the operator A is transformed into

$$\alpha(y,x')\left\{\frac{\partial^2}{\partial y^2}+\sum_{i,j=1}^n s^{ij}(y,x')\frac{\partial^2}{\partial x'_i\partial x'_j}+\sum_{j=1}^n s^j(y,x')\frac{\partial}{\partial x'_j}+b(y,x')\frac{\partial}{\partial y}+c(y,x')\right\}, \tag{2.1}$$

where $\alpha>0$ on $\overline{Q_{\delta_0}}$, $s^{ij}=s^{ji}$ and all the coefficients are real valued functions belonging to $C^\infty(\overline{Q_{\delta_0}})$.

Of course, it suffices to consider $\alpha=1$. This coordinate transformation (we write x in place of x') reduces A1 and A4 to

$$S_2(y,x,\xi)=\sum_{i,j=1}^n s^{ij}(y,x)\xi_i\xi_j\geqq 0 \qquad \text{for } (y,x,\xi)\in\overline{Q_{\delta_0}}\times(R^n\backslash\{0\}) \tag{2.2}$$

and

$$S_1^s(y,x,\xi)=\sum_{j=1}^n\left\{s^j(y,x)-\sum_{i=1}^n\frac{\partial s^{ij}}{\partial x_i}(y,x)\right\}\xi_j\neq 0 \tag{2.3}$$

$$\text{for } (y,x,\xi)\in\Sigma_{\delta_0}=\{(y,x,\xi)\in\overline{Q_{\delta_0}}\times(R^n\backslash\{0\})\mid S_2(y,x,\xi)=0\}.$$

Set $C_{(0)}^\infty([0,\delta)\times B_\delta)=\{u \mid u \text{ is a restriction of } w\in C_0^\infty((-\delta,\delta)\times B_\delta) \text{ to } [0,\delta)\times B_\delta\}$ for δ, $0<\delta\leqq\delta_0$. To prove Theorem 1 it is sufficient to show the following

Proposition. *Assume that the conditions* (2.2) *and* (2.3) *hold and that* $u\in L^2(Q_{\delta_0})$ *and* $f\in H^k(Q_{\delta_0})$ *satisfy the identity*

$$\iint_{Q_{\delta_0}} u\overline{A^t v}\,dydx=\iint_{Q_{\delta_0}} f\bar{v}\,dydx \qquad \textit{for all } v\in C_{(0)}^\infty([0,\delta_0)\times B_{\delta_0}) \tag{2.4}$$

with

$$v(0, x)=0.$$

Then there is a positive constant $\delta<\delta_0$ *such that* $u\in H^{k+1}(Q_\delta)$.

To show the proposition we prepare the following lemmas.

Set

$$T_r u(y, x)=\frac{1}{(2\pi)^n}\int_{R^n} e^{ix\cdot\xi}(1+|\xi|^2)^{r/2}\hat{u}(y, \xi)d\xi$$

for $u\in C^\infty_{(0)}([0, \delta)\times B_\delta)$ and $r\in R$, where $\hat{u}(y, \xi)$ is the Fourier transform of $u(y, x)$ with respect to x. Integration by parts gives

Lemma 1. *Assume that the conditions* (2.2) *and* (2.3) *hold. If* δ *is sufficiently small, then for any* $r\in R$, *there is a constant* C *independent of* u *such that*

$$\left\|T_r\frac{\partial^2 u}{\partial y^2}\right\|_0+\left\|T_r S_2\left(y, x, \frac{\partial}{\partial x}\right)u\right\|_0+\|T_{r+1}u\|_0\leqq C(\|T_r Au\|_0+\|T_r u\|_0) \tag{2.5}$$

for all $u\in C^\infty_{(0)}([0, \delta)\times B_\delta)$ *with* $u(0, x)=0$, *where* $\|\ \ \|_0$ *denotes the norm in* $L^2(R^{n+1}_+)$.

Set

$$S_2^{(l)}\left(y, x, \frac{\partial}{\partial x}\right)=\sum_{i=1}^n s^{il}(y, x)\frac{\partial}{\partial x_i}$$

and

$$S_{2(l)}\left(y, x, \frac{\partial}{\partial x}\right)=\sum_{i,j=1}^n \frac{\partial s^{ij}}{\partial x_l}(y, x)\frac{\partial^2}{\partial x_i\partial x_j}$$

for $l=1, 2, \cdots, n$. A sharp form of Gårding's inequality gives

Lemma 2. *Assume that the condition* (2.2) *holds. Then for any* $r\in R$, *there is a constant* C *independent of* u *such that*

$$\sum_{l=1}^n\left\|T_{r+(1/2)}S_2^{(l)}\left(y, x, \frac{\partial}{\partial x}\right)u\right\|_0+\sum_{l=1}^n\left\|T_{r-(1/2)}S_{2(l)}\left(y, x, \frac{\partial}{\partial x}\right)u\right\|_0 \tag{2.6}$$

$$\leqq C\left(\left\|T_r S_2\left(y, x, \frac{\partial}{\partial x}\right)u\right\|_0+\|T_{r+1}u\|_0\right)$$

for all $u\in C^\infty_{(0)}([0, \delta)\times B_\delta)$.

Once Lemmas 1 and 2 have been proved, the analogous reasoning to Hörmander [3] gives the proposition.

3. We shall remark on the boundary value problem of Neumann type. Let Ω be a bounded domain in $R^2_+=\{(y, x)\in R^2\,|\,y>0\}$ with C^∞ boundary $\partial\Omega$. For simplicity we shall assume that in a neighborhood U of the origin the boundary $\partial\Omega$ is described by the equation $y=\psi(x)$ with $\psi(x)\geqq 0$ and $\psi(0)=0$ and that the boundary $\partial\Omega$ doesn't touch the x axis outside of U. Let's consider the following boundary value problem:

[P′] $$(A-\alpha)u=\frac{\partial^2 u}{\partial y^2}+y\frac{\partial^2 u}{\partial x^2}+c\frac{\partial u}{\partial x}-\alpha u=f \qquad \text{in } \Omega,$$

$$\partial_\nu u\Big|_{\partial\Omega}=n_1\frac{\partial u}{\partial y}+n_2 y\frac{\partial u}{\partial x}\Big|_{\partial\Omega}=0 \qquad \text{with a positive constant } \alpha,$$

where c is a nonzero real constant and $n=(n_1, n_2)$ is the unit inner normal vector to $\partial\Omega$ (∂_ν denotes the conormal derivative corresponding to A).

We obtain

Theorem 2. *There is a constant $\alpha_0>0$ such that we have a weak solution $u \in L^2(\Omega)$ of [P′] with $f \in L^2(\Omega)$ if $\alpha>\alpha_0$. Moreover any weak solution $u \in L^2(\Omega)$ of [P′] with $f \in H^k(\Omega)$ belongs to $H^{k+1}(\Omega)$. In particular, if $f \in C^\infty(\bar{\Omega})$, then $u \in C^\infty(\bar{\Omega})$.*

Here $u \in L^2(\Omega)$ is called a weak solution of [P′] with $f \in L^2(\Omega)$ if the identity

$$(3.1) \quad \iint_\Omega u\overline{(A-\alpha)^t v}\, dydx = \iint_\Omega f\bar{v}\, dydx \qquad \text{holds for all } v \in C^\infty(\bar{\Omega})$$

with

$$(\partial_\nu - cn_2)v|_{\partial\Omega} = 0.$$

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