115. *The Gauss Map in Models*

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1. Introduction. Let N be an n-dimensional Riemannian manifold isometrically immersed into a Euclidean $(n+k)$-space E^{n+k} $(k\geqq 1)$ and $\mathcal{V}_E(N)$ be the unit normal bundle of N in E^{n+k}. Then the Gauss map of $\mathcal{V}_E(N)$ into the unit sphere about the origin of E^{n+k} was given by Chern and Lashof [1]. J. L. Weiner [5] gave a generalization of this map as follows: Let N be an isometrically immersed n-dimensional Riemannian manifold into a complete $(n+k)$-dimensional Riemannian manifold. Suppose that for a point p of N, N does not intersect the cut locus of p. The parallel displacement of $v\in\mathcal{V}_M(N)$ (=the unit normal bundle of N in M) along the shortest geodesic segment joining the foot point of v to p gives a mapping of $\mathcal{V}_M(N)$ into the unit sphere in the tangent space of M at p. This map is called the Gauss map on N based at p. R. Takagi [4] described an n-dimensional complete Riemannian N isometrically immersed into a Euclidean $(n+1)$-sphere S^{n+1} when the Gauss map on N based at a point S^{n+1} has constant rank. Furthermore, J. L. Weiner [5] showed similar results when the ambient space is a hyperbolic space of curvature -1 and also reproved Takagi's theorem in a simpler fashion. When the ambient space M is a model with a pole o, the cut locus of o is empty. So, for any isometrically immersed Riemannian manifold N into M, the Gauss map G_M on N based at o can be defined. In this note, we will study the Gauss map G_M and show the similar results to those of J. L. Weiner.

2. Preliminaries. Let (M, o) be an n-dimensional model with a pole o $(n\geqq 2)$ and $h:=\mathrm{Exp}_o: M_o\to M$ be the exponential map from the tangent space M_o at o of M onto M. Choosing an orthonormal basis $\{e_1, \cdots, e_n\}$ on M_o, let $\{y^1, \cdots, y^n\}$ be the normal coordinate system relative to this basis. Let g be the Riemannian metric on M. Then h^*g is a Riemannian metric on M_o and written by

$$h^*g = dr^2 + f(r)^2 d\Theta^2.$$

Here $d\Theta^2$ denotes the canonical metric on the unit sphere of M_o, r is the usual radial function on M_o and $f(r)$ is the C^∞ function on $[0, \infty)$ satisfying

$$f(0)=0,\ f'(0)=1,\ f(r)>0 \qquad \text{for } r>0.$$

3. Parallel displacements. For a tangent vector

$$A=\sum_{i=1}^{n} a^i(p)\partial/\partial y^i(p) \quad \text{at} \quad p=h\left(\sum_{i=1}^{n} y^i e_i\right)\in M,$$

the parallel displacement of A from p to o along the geodesic segment ζ joining o and p is denoted by $\Gamma(A)$. Then, by the use of Jacobi fields, we have

Lemma 1.

$$\Gamma(A)=\frac{f(r)}{r}\sum_{i=1}^{n} a^i(p)e_i+\left(1-\frac{f(r)}{r}\right)g(A,\dot{\zeta}(r))\sum_{i=1}^{n}\frac{y^i}{r}e_i$$

where $r=(\sum_{i=1}^{n}(y^i)^2)^{1/2}=$ *the distance between* o *and* p.

Remark 1. We note that $g(A,\dot{\zeta}(r))=\sum_{i=1}^{n} a^i(p)(y^i/r)$ by the Gauss Lemma.

Now, we define the new coordinate system on M by

$$x^i(p):=\frac{y^i(p)}{r}\exp\left(\int_1^r \frac{ds}{f(s)}\right) \qquad (i=1,\cdots,n)$$

for $p\in M$, where $r=(\sum_{i=1}^{n}(y^i(p))^2)^{1/2}=$ the distance between o and p. It is shown as follows that $\{x^1,\cdots,x^n\}$ is a coordinate system of M.

Lemma 2. *The map* $\Psi: M\to R^n$ *defined by* $\Psi:=u\circ h^{-1}$ *is one to one and* C^∞ *map, where* u *is the map* $M_o\to R^n$ *defined by*

$$u\left(\sum_{j=1}^{n} y^j e_j\right)=(x^1,\cdots,x^n)$$

and

$$x^i:=\frac{y^i}{r}\exp\left(\int_1^r\frac{ds}{f(s)}\right),\qquad r=\left(\sum_{i=1}^{n}(y^i)^2\right)^{1/2}.$$

Proof. Since $h^{-1}: M\to M_o$ is diffeomorphism, it is sufficient to show that u is one to one and C^∞ map. It is clear that u is one to one on M_o and C^∞ in $M_o-\{0\}$. Let

$$F\left(\sum_{j=1}^{n} y^j e_j\right):=(1/r)\exp\left(\int_1^r\frac{ds}{f(s)}\right).$$

Since

$$\partial F/\partial y^i=F\frac{y^i}{r^2}\left(\frac{r}{f(r)}-1\right)$$

and $\lim_{r\to 0}F$ is uniquely determined as to be a positive constant, we must show that each

$$\eta^i:=\frac{y^i}{r^2}\left(\frac{r}{f(r)}-1\right)$$

has a smooth extension across the origin. It is known ([2], [3]) that $f(r)=r+r^3l(r)$ and $l(r)$ has a smooth extension across the origin. Thus

$$\eta^i=\frac{y^i}{r^2}\left(\frac{r}{r+r^3l(r)}-1\right)=\frac{-y^il(r)}{1+r^2l(r)}$$

has a smooth extension across the origin.

Now we have the parallel translation in terms of the new coordinate system $\{x^i\}$ by Lemma 1 and Remark 1.

Lemma 3. *For a tangent vector* $A=\sum_{i=1}^{n} b^i(p)(\partial/\partial x^i)(p)$ *at* $p\in M$, *we have*

$$\Gamma(A)=\frac{F(0)f(r)}{\exp\left(\int_1^r \frac{ds}{f(s)}\right)} \sum_{i=1}^n b^i(p)\frac{\partial}{\partial x^i}(o)$$

where r *= the distance between* o *and* p, *and*

$$F(0)=\lim_{r\to 0}(1/r)\exp\left(\int_1^r \frac{ds}{f(s)}\right)=\lim_{r\to 0}(1/f(r))\exp\left(\int_1^r \frac{ds}{f(s)}\right).$$

Now let $\rho=\exp\left(\int_1^r \frac{ds}{f(s)}\right)$. Then $\rho=(\sum_{i=1}^n (x^i)^2)^{1/2}$ and so by Lemma 3, we have

Proposition. *Let* (M, o) *be an* n*-dimensional model with a pole* o. *Then*

(1) *If* $\exp\left(\int_1^\infty \frac{ds}{f(s)}\right)=\rho_0<\infty$, M *is isometric to*

$$D^n=\{(x^1,\cdots,x^n)\mid \sum_{i=1}^n (x^i)^2=\rho_0^2\}$$

with the Riemannian metric $\gamma(\rho)^2 g_0$ *where* g_0 *is the restriction of the canonical Euclidean metric on* R^n *to* D^n.

(2) *If* $\exp\left(\int_1^\infty \frac{ds}{f(s)}\right)=\infty$, M *is isometric to* R^n *with the Riemannian metric* $\gamma(\rho)^2 g_0$.

4. The Gauss map. Let N be an n-dimensional Riemannian manifold isometrically immersed in an $(n+k)$-dimensional model (M, o) with a pole o $(k\geqq 1)$. Then, by Proposition in § 3, N may be immersed (not isometrically in general) in the Euclidean space E^{n+k} of dimension $n+k$. Let G_E be the usual Gauss map in E^{n+k}. By Lemma 3, we have

Lemma 4. *Let* N *be an* n*-dimensional Riemannian manifold isometrically immersed in an* $(n+k)$*-dimensional model* (M, o). *Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{V}_E(N) & \xrightarrow{\ G_E\ } & S^{n+k-1}(1) \\ \bar{\gamma}\downarrow & & \downarrow \times 1/\gamma(0) \\ \mathcal{V}_M(N) & \xrightarrow{\ G_M\ } & S^{n+k-1}(1/\gamma(0)), \end{array}$$

where $\bar{\gamma}(v):=(1/\gamma(\rho))v$ *for a unit normal vector* v *at* $p=(x^1,\cdots,x^n)$, ρ *and* γ *are the same as in Proposition, and* $S^{n+k-1}(\alpha)$ *is the sphere about the origin in* E^{n+k} *of radius* α.

Corollary. *Since* $\bar{\gamma}:\mathcal{V}_E(N)\to\mathcal{V}_M(N)$ *is a diffeomorphism, the rank of* G_E *at* v *equals the rank of* G_M *at* $\bar{\gamma}(v)$ *for all* $v\in\mathcal{V}_E(N)$.

If N is orientable and $k=1$, we can identify N with a component of $\mathcal{V}_M(N)$ and also the corresponding component of $\mathcal{V}_E(N)$. Then $G_M: N\to$ the unit n-sphere about the origin in M_o is the Gauss map based at o and $G_E: N\to S^n(1)$ is the usual Gauss map in E^{n+1}.

Theorem. *Let* N *be an* n*-dimensional complete orientable Riemannian manifold isometrically immersed in an* $(n+1)$*-dimensional* *model* (M, o). *Suppose that* G_M *has constant rank* $n-m$ *on* N $(0 \leqq m \leqq n)$.

(1) *If* $m=0$ *and* N *is compact, then* N *is diffeomorphic to the* n-*sphere*.

(2) *If* $1 \leqq m \leqq n-1$, *then* N *is foliated by* m-*dimensional totally umbilic submanifolds*.

(3) *If* $m=n$, *then* N *is a totally umbilic hypersurface*.

Proof. (1) is clear.

Since the rank of G_M equals the rank of G_E by corollary, N is foliated by m-dimensional planes L^m in E^{n+1} intersected with M by Lemma 2 of [1]. For each L^m, $L^m \cap M$ with the induced metric from M is a totally umbilic submanifold. Thus (2) and (3) are verified.

Remark 2. If M is the hyperbolic space, each totally umbilic submanifold in Theorem is a hyperbolic space of a certain constant curvature (see [5]).

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