109. On a Question Posed by Huckaba-Papick. II

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1. Introduction. This is a continuation of [5]. As in the introduction of [5], let R be an integral domain with the quotient field K, and let x be an indeterminate. By $c(f)$ we denote the ideal of R generated by the coefficients of f for an element f of $R[x]$. We denote the subset $\{f \in R[x]; c(f)^{-1}=R\}$ of $R[x]$ by U, where $c(f)^{-1}=\{a \in K; ac(f) \subset R\}$. Let $\mathcal{P}(R)$ be the set of prime ideals of R which are minimal prime ideals over $(a : b)$ for some elements a, b of R. Huckaba-Papick ([2]) posed the following questions:

Questions ([2, Remark (3.4)]). (a) If R_P is a valuation ring for each $P \in \mathcal{P}(R)$, is $R[x]_U$ a Prüfer ring?

(b-1) If $R[x]_U$ is a Bezout ring, are the prime ideals of $R[x]_U$ extended from prime ideals of R?

(b-2) If $R[x]_U$ is a Prüfer ring, are the prime ideals of $R[x]_U$ extended from prime ideals of R?

(c) If $R[x]_U$ is a Prüfer ring, is it a Bezout ring?

In [4], we answered to the question (b-1) in the affirmative, and showed that questions (b-2) and (c) are equivalent. In [5], we answered to the question (c) in the affirmative. The purpose of this paper is to give a negative answer to the question (a) in proving the following result:

Proposition. *There exists an integral domain R such that R_P is a valuation ring for each $P \in \mathcal{P}(R)$ and that $R[x]_U$ is not a Prüfer ring.*

2. Proof of Proposition. Lemma 1. *If $R[x]_U$ is a Prüfer ring, then the prime ideals of $R[x]_U$ are extended from prime ideals of R.*

Proof. By [5, Theorem 1], $R[x]_U$ is a Bezout ring. By [4, Theorem 1], the prime ideals of $R[x]_U$ are extended from prime ideals of R.

Throughout the rest of the paper, we denote by R the integral domain $Z[2u, 2u^2, 2u^3, \cdots]$ where u is an indeterminate over Z, and by K the quotient field of R (cf. [1, § 25, Exercise 21]).

Lemma 2 ([3, II, a part of Example 2]). (1) *The maximal ideal $M=(2, 2u, 2u^2, \cdots)$ of R is a minimal prime ideal over the principal ideal* (2).

(2) *R_M is a valuation ring.*

(3) *M is the only maximal ideal of R containing* 2.

(4) *R is integrally closed.*

(5) *R is 2-(Krull)-dimensional.*

Lemma 3. (1) *The quotient ring of R with respect to the multiplicative subset of R generated by 2 is the subring $\boldsymbol{Z}[1/2, u]$ of $\boldsymbol{Q}[u]$.* ($\boldsymbol{Q}$ is the field of rational numbers.)

(2) *$\boldsymbol{Z}[1/2, u]$ is a unique factorization ring.*

(3) *Let p be an odd prime number. Then (p) is a prime ideal of R.*

Proof. (1) The proof is obvious. (2) Since $\boldsymbol{Z}[1/2]$ is a quotient ring of $\boldsymbol{Z}$, it is a unique factorization ring. Since $\boldsymbol{Z}[1/2, u]$ is a polynomial ring over $\boldsymbol{Z}[1/2]$, it is a unique factorization ring. (3) Let $r_1 r_2 \in (p)$ for elements $r_1, r_2 \in R$. Since $p\boldsymbol{Z}[u]$ is a prime ideal of $\boldsymbol{Z}[u]$, we see that either r_1 or r_2, say r_1, belongs to $p\boldsymbol{Z}[u]$. We have $r_1 = pF$ for some $F \in \boldsymbol{Z}[u]$. Since p is an odd number, it follows $F \in R$. Hence (p) is a prime ideal of R.

Lemma 4. *Let M be a prime ideal of R of height 2, containing an odd prime number p. Then we have $M \notin \mathcal{P}(R)$.*

Proof. We have $M \not\ni 2$. By Lemma 3, (1), $M\boldsymbol{Z}[1/2, u]$ is a prime ideal of $\boldsymbol{Z}[1/2, u]$ of height 2. By Lemma 3, (2), we have $M\boldsymbol{Z}[1/2, u] \supsetneqq p\boldsymbol{Z}[1/2, u]$. We choose $r \in M-(p)$, and set $f = p + rx$. Let $k \in c(f)^{-1}$ for an element $k \neq 0$ of K. We have $pk = r_1$ and $rk = r_2$ for $r_1, r_2 \in R$. Hence $r_1 r = p r_2$. By Lemma 3, (3), we have $r_1 \in (p)$. It follows that $k \in R$, and hence $c(f)^{-1} = R$. Since $f \in MR[x]$, we have $M \notin \mathcal{P}(R)$ by [6, Theorem E].

Lemma 5. *R_P is a valuation ring for each $P \in \mathcal{P}(R)$.*

Proof. Let M be a maximal ideal of R containing P. By Lemma 2, (3), we have the following three cases: (1) $M = (2, 2u, 2u^2, \cdots)$, (2) $M \cap \boldsymbol{Z} = 0$, and (3) M contains an odd prime number p. Case (1): R_P is a quotient ring of R_M. Hence R_P is a valuation ring by Lemma 2, (2). Case (2): R_P is a quotient ring of $\boldsymbol{Q}[u]$ with respect to its prime ideal $P\boldsymbol{Q}[u]$. It follows that R_P is a valuation ring. Case (3): If height $P > 1$, then we have height $P = 2$ and $P = M$ by Lemma 2, (5). By Lemma 4, it follows $P \notin \mathcal{P}(R)$, which is a contradiction. Hence height $P \leq 1$. By Lemma 3, (1), we see that $P\boldsymbol{Z}[1/2, u]$ is a prime ideal of $\boldsymbol{Z}[1/2, u]$ of height ≤ 1. By Lemma 3, (2), $\boldsymbol{Z}[1/2, u]_{P\boldsymbol{Z}[1/2, u]}$ is a valuation ring. Since $R_P = \boldsymbol{Z}[1/2, u]_{P\boldsymbol{Z}[1/2, u]}$, R_P is a valuation ring.

Lemma 6. *$R[x]_U$ is not a Prüfer ring.*

Proof. R in an integrally closed ring (Lemma 2, (4)). We set $M = (2, 2u, 2u^2, \cdots)$, and set $f = 2 + 2ux$. By Lemma 2, (1), we have $M \in \mathcal{P}(R)$. $fK[x]$ is a prime ideal of $K[x]$. We set $fK[x] \cap R[x] = Q$. By [6, Theorem B], we have $Q = c(f)^{-1} f R[x]$. Let $k \in c(f)^{-1}$ for an element $k \neq 0$ of K. We have $2k = r_1$ and $2uk = r_2$ for $r_1, r_2 \in R$. It follows $ur_1 = r_2$, and hence $r_1 \in M$. Therefore we have $k \in \boldsymbol{Z}[u]$ and kf

$\in MR[x]$. We have shown $Q \subset MR[x]$. By [6, Theorem E], we have $Q \cap U = \emptyset$. Hence $QR[x]_U \cap R = Q \cap R$. Since $Q \cap R = 0$, it follows $QR[x]_U \supsetneq (QR[x]_U \cap R)R[x]_U$. By Lemma 1, $R[x]_U$ is not a Prüfer ring.

Lemmas 5 and 6 complete the proof of Proposition.

References

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